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Weighted Wireless Link Scheduling without Information of Positions And Interference/Communication Radii

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***Abstract*—Link scheduling is a fundamental design issue in multihop wireless networks. All existing link scheduling algorithms require the precise information of the positions, and/or communication/interference radii of all nodes. For practical networks, it is not only difficult or expensive to obtain these parameters, but also often impossible to get their precise values. The link scheduling determined by the imprecise values of these parameters may fail to guarantee the same approximation bounds of the link scheduling determined by precise values. Therefore, the existing link scheduling algorithms lack performance robustness. In this paper, we propose a robust link scheduling, which can be easily computed with only the information on whether a given pair of links have conflict or not and therefore is robust. In addition, our link scheduling does not compromise the approximation bound and indeed sometimes can achieve better approximation bound. Particularly, under the 802.11 interference model, its approximation bound is 16 in general and 6 with uniform interference radii, an improvement over the respective best-known approximation bounds 23 and 7.**

***Index Terms*—Link scheduling, interference, robustness, latency, approximation algorithm.**

I. Introduction

Link scheduling is a fundamental design issue in multihop wireless networks. A multihop wireless network $\mathbf{N}$ is specified, in its most general format, by a triple $(V, A, \mathcal{I})$, where V is the set of networking nodes, A is the set of communication links among V, and $\mathcal{I}$ is the collection of sets of independent (or conflict-free) links in A specified implicitly by an interference model. The communication topology of $\mathbf{N}$ is the digraph (V, A). A *(fractional) link schedule* Π in $\mathbf{N}$ is a set

$$\{(I_j, \ell_j) \in \mathcal{I} \times \mathbb{R}_+ : 1 \leq j \leq k\};$$

the two values k and $\sum_{j=1}^{k} \ell_j$ are referred to as the *size* and *length* (or *latency*) of Π respectively. Π is said to be the fractional link schedule for a link demand $d \in \mathbb{R}_+^A$ if for each link $a \in A$,

$$d(a) = \sum_{j=1}^{k} \ell_j |I_j \cap \{a\}|.$$

The problem **Shortest Fractional Link Schedule** (**SFLS**) seeks a shortest (fractional) link schedule for a given a set of communication links together with their demands.

The problem **SFLS** under either the 802.11 interference model or the protocol interference model has been well-studied recently. An instance of a network $\mathbf{N}$ under either the 802.11 interference model or the protocol interference model is specified by a finite planar set V of nodes together with a communication radius function $r \in \mathbb{R}_+^V$ and an interference radius function $\rho \in \mathbb{R}_+^V$ satisfying that $\rho \geq r$. The communication (respectively, interference) range of a node $v \in V$ is the disk centered at v of radius $r(v)$ (respectively, $\rho(v)$). The set A of communication links and the collection $\mathcal{I}$ of independent sets of links are defined as follows:

- 802.11 interference model: A consists of all pairs (u, v) satisfying that u and v are within each other's communications ranges. Two links in A conflict with each other if and only if at least one link has an endpoint lying in the interference range of some endpoint of the other link. A set I of links in A are independent (i.e., $I \in \mathcal{I}$) if all links in I are mutually conflict-free.
- Protocol interference model: A consists of all pairs (u, v) satisfying that u and v are within each other's communications ranges. Two links in A conflict with each other if and only if the receiving end of at least one link lies in the interference range of the transmitting end of the other link. A set I of links in A are independent (i.e., $I \in \mathcal{I}$) if all links in I are mutually conflict-free.

Under either of these two interference models, the problem **SFLS** is NP-hard even when all nodes have uniform (and fixed) communication radii and uniform (and fixed) interference radii and the positions of all nodes are available [7]. On the other hand, it admits a polynomial-time approximation scheme (PTAS) [7] under the 802.11 interference model or under the protocol interference model with some additional mild conditions. In other words, for any fixed $\varepsilon > 0$, it has polynomial-time (depending on ε) $(1 + \varepsilon)$-approximation algorithm. Such PTAS is of theoretical interest only and is quite infeasible practically. A series of works [1] [2] [6] [8] have been devoted to practical constant approximation algorithms. However, all these algorithms have *super-exponential* running-time in the worst case as observed in [7]. Wan [7] then developed approximation algorithms which not only have truly polynomial running time but also have improved approximation bounds.

Under either the 802.11 interference model or the protocol interference model, all the known algorithms for **SFLS** [1] [2] [6] [8] [7] require either the positions of all nodes or the interference/communication radii of all nodes. For practical networks, these parameters are difficult or very expensive to be obtained. In addition, we can only hope for estimated values rather than the precise values of these parameters. However, the link scheduling determined by the estimated values of these parameters may no longer be able to guarantee the proved approximation bounds. In this perspective, all the known link scheduling algorithms lack performance robustness. This drawback of the known algorithms raises naturally the question on whether there exists a practical approximation algorithm for **SFLS** which does not require any information of the positions and the interference/communication radii of all nodes but still achieves the same or even better approximation bounds.

In this paper, we give a positive answer to the above question by developing a simple and efficient approximation algorithm for **SFLS** which requires only the "topological" information on whether a given pair of links have conflict or not, and the information on the link demands. Its implementation is independent of the interference model. In addition, it does not compromise the approximation bound. Indeed, its approximation bound cannot be worse than any approximation bound obtained in [7]. Specifically, under the protocol interference model, its approximation bound is at least the same as the one in [7]; under the 802.11 interference model, its approximation bound is 16 in general and 6 with uniform interference radii, an improvement over the respective approximation bounds 23 and 7 derived in [7].

The following standard notations and terms are adopted throughout this paper. Let S be a finite subset. For any real function $f \in \mathbb{R}^S$ and any subset $S' \subseteq S$, $f(S')$ denotes $\sum_{e \in S'} f(e)$. Let $G = (V, E)$ be an undirected graph. For any subset U of V, $G[U]$ denotes the subgraph of G induced by U. For any node $v \in V$, $N_G(v)$ denotes the set of neighbors of v in G, and $N_G[v]$ denotes $\{v\} \cup N_G(v)$. For any $d \in \mathbb{R}_+^V$, $d(N_G(v))$ is referred to as the weighted degree of v in (G, d), and $d(N_G[v])$ is referred to as the closed weighted degree (CWD) of v in (G, d). Consider a vertex ordering $\prec$ of V. For each $v \in V$, $N_G^{\prec}(v)$ denotes the set of neighbors of v preceding v in the ordering $\prec$, and $N_G^{\prec}[v]$ denotes $\{v\} \cup N_G^{\prec}(v)$. An *orientation* of G is a digraph obtained from G by imposing an orientation on each edge of G. Suppose that H is digraph with vertex set V. For any subset U of V, $H[U]$ denotes the subgraph of H induced by U. For each $v \in V$, $N_H^{in}(v)$ denotes the set of in-neighbors of v in H, and $N_H^{in}[v]$ denotes $\{v\} \cup N_H^{in}(v)$; similarly, $N_H^{out}(v)$ denotes the set of out-neighbors of v in H, and $N_H^{out}[v]$ denotes $\{v\} \cup N_H^{out}(v)$. The Euclidean distance between two points u and v in the plane is denoted by $\|uv\|$.

The remaining of this paper is organized as follows. In Section II, we present a unified framework for first-fit fractional weighted coloring in arbitray graphs. Following such general framework, we develop the design and analysis of a first-fit fractional weighted link scheduling algorithm in multihop wireless networks in Section III. Finally, we conclude this paper in Section IV.

II. Fractional Weighted Coloring

Let $G = (V, E)$ be an undirected graph. A subset I of V is an *independent set* (IS) of G if no two nodes in I are adjacent. If I is a independent set of G but no proper superset of I is a independent set of G, then I is called a *maximal independent set* of G. Any node ordering $\langle v_1, v_2, \cdots, v_n \rangle$ of V induces a maximal IS I in the following first-fit manner: Initially, $I = \{v_1\}$. For $i = 2$ up to n, add v_i to I if v_i is not adjacent to any node in I. An independent set of the largest size is called a *maximum independent set*. Let $\mathcal{I}$ be the collection of all independent sets of G. The *independence number* of G, denoted by $\alpha(G)$, is defined to be $\max_{I \in \mathcal{I}} |I|$. For any node v, the value $\alpha(G[N_G[v]])$ is referred to as the *local independence number* (LIN) of v in G. For any $d \in \mathbb{R}_+^V$, a *fractional (weighted) coloring* of (G, d) is a set of k pairs $(I_j, \ell_j) \in \mathcal{I} \times \mathbb{R}_+$ for $1 \leq j \leq k$ satisfying that for each $v \in V$,

$$\sum_{j=1}^{k} \ell_j |I_j \cap \{v\}| = d(v).$$

The two values k and $\sum_{j=1}^{k} \ell_j$ are referred to as the number and length of the coloring respectively. The *fractional chromatic number* $\chi_f(G, d)$ of (G, d) is defined as the minimum length of all fractional colorings of (G, d).

A first-fit fractional weighted coloring algorithm was developed in [7], which is described as follows. Consider a vertex ordering $\prec$ of G. For any $d \in \mathbb{R}_+^V$, this algorithm produces a fractional coloring Π of (G, d) in the following iterative manner. Initially, Π is empty. In each iteration, let V' be the subset of vertices $v \in V$ with $d(v) > 0$. Compute a maximal set I of independent vertices in V' in the first-fit manner in the ordering $\prec$ restricted to V'. Let $\ell = \min_{v \in I} d(v)$, and add (I, ℓ) to Π. For each $v \in I$, replace $d(v)$ by $d(v) - \ell$. Repeat this iteration until $d = 0$. The output Π is referred to as the *first-fit fractional coloring of* (G, d) *in the ordering* $\prec$. Since in each iteration the number of vertices v with $d(v) > 0$ strictly decreases, the number of iterations, or the number of colors, is bounded by n. In addition, if d is integral then the output coloring Π is also integral. The next theorem was proved in [7], which gives an upper bound on the length of Π.

Theorem 1: The first-fit fractional coloring of (G, d) in a vertex ordering $\prec$ of G has length at most $\max_{v \in V} d(N_G^{\prec}[v])$.

For any vertex ordering $\prec$, the value $\max_{v\in V} d\left(N_G^{\preceq}[v]\right)$ is referred to as its *closed d-inductivity*. The smallest closed d-inductivity of all possible vertex ordering is called the *closed d-inductivity* of G,which is denoted by $\delta^*(G,d)$. A natural question is whether a vertex ordering of the smallest closed d-inductivity can be computed in polynomial time. The answer to this question is positive. In the next, we describe a special vertex ordering called *smallest-CWD-last ordering* and show that it achieves the smallest d-inductivity. It is produced iteratively as follows: Initialize G' to G. For $i=n$ down to 1, let v_i be a vertex of the smallest closed weighted degree in G' and delete v_i from G'. Then the ordering $\langle v_1, v_2, \cdots, v_n\rangle$ is a smallest-CWD-last ordering.

Theorem 2: The smallest-CWD-last ordering achieves the closed d-inductivity $\delta^*(G,d)$ and

$$\delta^*(G,d)=\max_{U\subseteq V}\min_{u\in U} d\left(N_{G[U]}[u]\right).$$

Proof: Denote

$$\overline{\delta}^*(G,d)=\max_{U\subseteq V}\min_{u\in U} d\left(N_{G[U]}[u]\right).$$

We first show that the closed d-inductivity of any vertex ordering $\prec$ is at least $\overline{\delta}^*(G,d)$. Let $U\subseteq V$ be such that

$$\overline{\delta}^*(G,d)=\min_{u\in U} d\left(N_{G[U]}[u]\right)$$

Let u be the last vertex in this ordering $\prec$ such that $u\in U$. Then,

$$N_G^{\preceq}[u]\supseteq N_{G[U]}[u],$$

which implies

$$\max_{v\in V} d\left(N_G^{\preceq}[v]\right)\geq d\left(N_G^{\preceq}[u]\right)\geq d\left(N_{G[U]}[u]\right)\geq \overline{\delta}^*(G,d).$$

Next, let $\prec$ be the smallest-CWD-last ordering $\langle v_1, v_2, \cdots, v_n\rangle$ and we show that its closed d-inductivity is at most $\overline{\delta}^*(G,d)$. For any $1\leq i\leq n$, let $V_i=\{v_1, v_2, \cdots, v_i\}$. Then,

$$N_G^{\preceq}[v_i]=N_{G[V_i]}[v_i].$$

By the selection criteria of v_i, we have

$$d\left(N_G^{\preceq}[v_i]\right)=\min_{u\in V_i} d\left(N_{G[V_i]}[u]\right)\leq \overline{\delta}^*(G,d).$$

Hence,

$$\max_{1\leq i\leq n} d\left(N_G^{\preceq}[v_i]\right)\leq \overline{\delta}^*(G,d).$$

Thus, the closed d-inductivity of $\langle v_1, v_2, \cdots, v_n\rangle$ is at most $\overline{\delta}^*(G,d)$.

Therefore, the smallest-CWD-last ordering achieves the smallest closed d-inductivity $\overline{\delta}^*(G,d)$ among all vertex orderings. So, $\delta^*(G,d)=\overline{\delta}^*(G,d)$ and the theorem follows. ∎

Next, we present two upper bounds on the closed d-inductivity of G. We will need the following property of node-weighted digraphs.

Lemma 3: Let H be a digraph with vertex set V. Then for any $d\in\mathbb{R}_+^V$, there exists at least one node $u\in V$ satisfying $d\left(N_H^{in}(u)\right)\geq d\left(N_H^{out}(u)\right)$.

Proof: Let

$$U=\{v\in V: d(v)>0\}.$$

The lemma holds trivially if U is empty. So assume that U is nonempty. For any link (u,v) in $H[U]$, we define its weight $c(u,v)=d(u)\,d(v)$. Since both $\sum_{u\in U}\sum_{v\in N_{H[U]}^{in}(u)} c(u,v)$ and $\sum_{u\in U}\sum_{v\in N_{H[U]}^{out}(u)} c(u,v)$ are equal to the total weight of all links in $H[U]$, we have

$$\sum_{u\in U}\sum_{v\in N_{H[U]}^{in}(u)} c(u,v)=\sum_{u\in U}\sum_{v\in N_{H[U]}^{out}(u)} c(u,v),$$

and consequently

$$\sum_{u\in U}\left(\sum_{v\in N_{H[U]}^{in}(u)} c(u,v)-\sum_{v\in N_{H[U]}^{out}(u)} c(u,v)\right)=0.$$

So, there is at least one node $u\in U$ satisfying that

$$\sum_{v\in N_{H[U]}^{in}(u)} c(u,v)\geq \sum_{v\in N_{H[U]}^{out}(u)} c(u,v).$$

Note that

$$\begin{aligned}\sum_{v\in N_{H[U]}^{in}(u)} c(u,v)&=\sum_{v\in N_{H[U]}^{in}(u)} d(u)\,d(v)\\&=d(u)\sum_{v\in N_{H[U]}^{in}(u)} d(v)=d(u)\,d\left(N_{H[U]}^{in}(u)\right),\end{aligned}$$

and similarly,

$$\sum_{v\in N_{H[U]}^{out}(u)} c(u,v)=d(u)\,d\left(N_{H[U]}^{out}(u)\right).$$

Therefore,

$$d(u)\,d\left(N_{H[U]}^{in}(u)\right)\geq d(u)\,d\left(N_{H[U]}^{out}(u)\right).$$

As $d(u)>0$, we have

$$d\left(N_{H[U]}^{in}(u)\right)\geq d\left(N_{H[U]}^{out}(u)\right).$$

Since

$$d\left(N_H^{in}(u)\right)=d\left(N_{H[U]}^{in}(u)\right),$$
$$d\left(N_H^{out}(u)\right)=d\left(N_{H[U]}^{out}(u)\right),$$

we have

$$d\left(N_H^{in}(u)\right)\geq d\left(N_H^{out}(u)\right).$$

Thus, the lemma holds. ∎

For any vertex ordering $\prec$ of G, its *backward local independence number* (BLIN) in G is defined to be

$$\max_{v\in V}\alpha\left(G\left[N_G^{\preceq}[v]\right]\right).$$

The *backward local independence number* (BLIN) of G, denoted by $\alpha^*(G)$, is defined as the smallest BLIN of all vertex ordering. For any orientation H of G, its *inward local independence number* (ILIN) in G is defined to be

$$\max_{v\in V} \alpha\left(G\left[N_H^{in}[v]\right]\right).$$

The *inward local independence number* (ILIN) of G, denoted by $\beta^*(G)$, is defined as the smallest ILIN of all possible orientations of G.

Theorem 4: For any $d \in \mathbb{R}_+^V$,

$$\delta^*(G,d) \le \min\{\alpha^*(G), 2\beta^*(G)\}\chi_f(G,d).$$

Proof: We first prove that

$$\delta^*(G,d) \le \alpha^*(G)\chi_f(G,d).$$

Suppose that $\prec$ is a vertex ordering whose BLIN is $\alpha^*(G)$ and

$$\{(I_j,\ell_j) \in \mathcal{I}\times\mathbb{R}_+ : 1 \le j \le k\}$$

is a shortest fractional coloring of (G,d). For any vertex $v \in V$, we have

$$\begin{aligned}
d\left(N_G^{\preceq}[v]\right) &= \sum_{u\in N_G^{\preceq}[v]} d(u) \\
&= \sum_{u\in N_G^{\preceq}[v]} \sum_{j=1}^{k} \ell_j \left|I_j \cap \{u\}\right| \\
&= \sum_{j=1}^{k} \sum_{u\in N_G^{\preceq}[v]} \ell_j \left|I_j \cap \{u\}\right| \\
&= \sum_{j=1}^{k} \ell_j \left|I_j \cap N_G^{\preceq}[v]\right| \\
&\le \sum_{j=1}^{k} \ell_j \alpha\left(G\left[N_G^{\preceq}[v]\right]\right) \\
&= \alpha\left(G\left[N_G^{\preceq}[v]\right]\right)\sum_{j=1}^{k} \ell_j \\
&= \alpha\left(G\left[N_G^{\preceq}[v]\right]\right)\chi_f(G,d) \\
&\le \alpha^*(G)\chi_f(G,d).
\end{aligned}$$

Therefore, the closed d-inductivity of any vertex ordering $\prec$ is at most $\alpha^*(G)\chi_f(G,d)$, and hence

$$\delta^*(G,d) \le \max_{v\in V} d\left(N_G^{\preceq}[v]\right).$$

Next, we prove that

$$\delta^*(G,d) \le 2\beta^*(G)\chi_f(G,d).$$

Suppose that H is an orientation of G whose ILIN is $\beta^*(G)$. We show that

$$\delta^*(G,d) \le 2\max_{v\in V} d\left(N_H^{in}[v]\right) \le 2\beta^*(G)\chi_f(G,d).$$

We begin with the first inequality. By Lemma 3, for any $U \subseteq V$, $H[U]$ contains at least one node u such that

$$d\left(N_{H[U]}^{in}(u)\right) \ge d\left(N_{H[U]}^{out}(u)\right).$$

Thus,

$$\begin{aligned}
d\left(N_{G[U]}[u]\right) &= d(u) + d\left(N_{H[U]}^{in}(u)\right) + d\left(N_{H[U]}^{out}(u)\right) \\
&\le d(u) + 2d\left(N_{H[U]}^{in}(u)\right) \\
&\le d(u) + 2d\left(N_H^{in}(u)\right) \\
&\le 2d\left(N_H^{in}[u]\right) \\
&\le 2\max_{v\in V} d\left(N_H^{in}[v]\right).
\end{aligned}$$

By Theorem 2, the first inequality holds. We move on to the second inequality. Suppose that

$$\{(I_j,\ell_j) \in \mathcal{I}\times\mathbb{R}_+ : 1 \le j \le k\}$$

is a shortest fractional coloring of (G,d). For any vertex $v \in V$, we have

$$\begin{aligned}
d\left(N_H^{in}[v]\right) &= \sum_{u\in N_H^{in}[v]} d(u) \\
&= \sum_{u\in N_H^{in}[v]} \sum_{j=1}^{k} \ell_j \left|I_j \cap \{u\}\right| \\
&= \sum_{j=1}^{k} \sum_{u\in N_H^{in}[v]} \ell_j \left|I_j \cap \{u\}\right| \\
&= \sum_{j=1}^{k} \ell_j \left|I_j \cap N_H^{in}[v]\right| \\
&\le \sum_{j=1}^{k} \ell_j \alpha\left(G\left[N_H^{in}[v]\right]\right) \\
&= \alpha\left(G\left[N_H^{in}[v]\right]\right)\sum_{j=1}^{k} \ell_j \\
&= \alpha\left(G\left[N_H^{in}[v]\right]\right)\chi_f(G,d) \\
&\le \beta^*(G)\chi_f(G,d).
\end{aligned}$$

Thus, the second inequality holds. ■

The above three theorems immediately imply the following general approximation bound on first-fit fractional coloring in the smallest-CWD-last ordering.

Corollary 5: The approximation ratio of first-fit fractional coloring in the smallest-CWD-last ordering is at most $\min\{\alpha^*(G), 2\beta^*(G)\}$.

III. Fractional Weighted Link Scheduling

Consider a multihop wireless network $(V,A,\mathcal{I})$ under either the 802.11 interference model or the protocol interference model. The *conflict graph* of A, denoted by G, is defined to be the undirected graph on A in which two links are adjacent if and only if they conflict with each other. Then, $\mathcal{I}$ is the essentially the collection of the independent sets in G; and for any link demand $d \in \mathbb{R}_+^A$, a (shortest) fractional link schedule for d corresponds to a (minimum) fractional coloring of (G, d). Therefore, we can compute a fractional link schedule for d in two steps. In the first step, we compute the smallest-CWD-last ordering of A using the iterative algorithm described in the previous section. In the second step, we compute a link schedule of (G, d) by using the first-fit fractional weighted coloring algorithm [7] also described in the previous section. Such algorithm is referred to as *first-fit fractional link scheduling in the smallest-CWD-last ordering*. Note that both steps require only the topological information on whether a given pair of links have conflict or not, and the information on the link demands. No information about the positions or the interference/communication radii of the nodes is needed.

In the next, we derive the approximation bound of the first-fit fractional link scheduling in the smallest-CWD-last ordering. We shall establish the following upper bounds on the BLIN and/or the ILIN of the conflict graph G.

Lemma 6: The following statements are true.

1) Under the protocol interference model, if the interference radius of each node is at least c times its communication radius for some $c > 1$, then
$$\beta^* (G) \leq \left\lceil \pi / \arcsin \frac{c-1}{2c} \right\rceil - 1.$$
2) Under the 802.11 interference model with uniform interference radii, $\alpha^* (G) \leq 6$.
3) Under the 802.11 interference model, $\beta^* (G) \leq 8$.

The above lemma together with Corollary 5 implies immediately the following approximation bounds on the first-fit fractional link scheduling in the smallest-CWD-last ordering.

Theorem 7: The first-fit fractional link scheduling in the smallest-CWD-last ordering has the following approximation bounds:

1) Under the protocol interference model, if the interference radius of each node is at least c times its communication radius for some $c > 1$, its approximation bound is at most $2 \left(\left\lceil \pi / \arcsin \frac{c-1}{2c} \right\rceil - 1 \right)$.
2) Under the 802.11 interference model with uniform interference radii, its approximation bound is at most 6.
3) Under the 802.11 interference model, its approximation bound is at most 16.

Note that under the protocol interference model, our approximation bound is the same as the best-known one obtained in [7]; under the 802.11 interference model with uniform interference radii, our approximation bound 6 is better than the best-known approximation bound 7 obtained in [7]; under the 802.11 interference model with arbitrary interference radii, our approximation bound 16 is better than the best-known approximation bound 23 obtained in [7]. Therefore, even without the information of the positions and the interference/communication radii of the nodes, we are still able to achieve the same or better approximation bounds.

The remaining of this section is devoted to the proof of Lemma 6. The first part and the second part of Lemma 6 can be easily proved. For the protocol interference model, we consider the following orientation H of G presented in [7]. For any pair of conflicting links $a_1 = (u_1, v_1)$ and $a_2 = (u_2 v_2)$, if v_1 is within the interference range of u_2 and v_2 is within the interference range of u_1, take an arbitrary orientation; otherwise, if v_1 is within the interference range of u_2, take the orientation from a_2 to a_1; otherwise, take the orientation from a_1 to a_2. It was shown in [7] that the ILIN of H is at most $\left\lceil \pi / \arcsin \frac{c-1}{2c} \right\rceil - 1$. Thus, the first part of Lemma 6 holds.

For the 802.11 interference model with uniform interference radii, we consider the lexicographic link ordering which sorts all links in the lexicographic order of their left endpoints. The Proposition 5 in [3] implies that for each link a, among all links which interfere with and precede the link a, at most 6 of them are independent. Hence, the BLIN of the lexicographic link ordering is at most 6. So, the second part of Lemma 6 holds.

We proceed to prove the third part of Lemma 6 by constructing an orientation H of G whose ILIN is at most 8 under the 802.11 interference model with arbitrary interference radii. For each node v, we use $\Gamma(v)$ to denote the set of nodes w satisfying that
$$\rho(w) \geq \max \{\rho(v), \|vw\|\}.$$
Consider a link $a = (u, v) \in A$ with $\rho(u) \geq \rho(v)$. It is said to be *dominated* by another link $a' \in A$ if a' has an endpoint w satisfying that either $w \in \Gamma(u)$, or $w \in \Gamma(v)$ and $\|uw\| > \max \{\rho(w), \rho(u)\}$. If a is dominated by a link a', then a' is said to *dominate* a. The lemma below establishes the existence of the domination relationship between two conflicting links.

Lemma 8: Suppose that a and a' are two conflicting links in A. Then, at least one of them dominates the other.

Proof: Consider two conflicting links $a = (u, v)$ and $a' = (u', v')$. By symmetry, we assume that $\rho(u) \geq \rho(v)$, $\rho(u') \geq \rho(v')$, and $\rho(u') \geq \rho(u)$. Then, neither endpoint of a' is in $\Gamma(u)$, for otherwise a' dominates a. So,
$$\rho(u') < \max \{\rho(u), \|uu'\|\}, \tag{1}$$
$$\rho(v') < \max \{\rho(u), \|uv'\|\}. \tag{2}$$
Since $\rho(u') \geq \rho(u)$, the inequality (1) implies that
$$\rho(u) \leq \rho(u') < \|uu'\|,$$
and hence
$$\|uu'\| > \max \{\rho(u), \rho(u')\}. \tag{3}$$
Thus, $u \notin \Gamma(v')$, for otherwise a dominates a' by the inequality (3). So,
$$\rho(u) < \max \{\rho(v'), \|uv'\|\}. \tag{4}$$
From the two inequalities the inequality (2) and (4), we have
$$\begin{aligned} &\max \{\rho(u), \rho(v')\} \\ &< \max \{\max \{\rho(v'), \|uv'\|\}, \max \{\rho(u), \|uv'\|\}\} \\ &= \max \{\rho(u), \rho(v'), \|uv'\|\}, \end{aligned}$$

which implies that

$$\|uv'\| > \max\{\rho(u), \rho(v')\}. \quad (5)$$

Consequently, neither endpoint of a' is in $\Gamma(v)$, for otherwise a' dominates a by the two inequalities (3) and (5). So,

$$\rho(u') < \max\{\rho(v), \|vu'\|\}, \quad (6)$$
$$\rho(v') < \max\{\rho(v), \|vv'\|\}. \quad (7)$$

Since $\rho(u') \geq \rho(u) \geq \rho(v)$, the inequality (6) implies that we have

$$\rho(v) \leq \rho(u') < \|vu'\|,$$

and hence

$$\|vu'\| > \max\{\rho(v), \rho(u')\}. \quad (8)$$

Thus, $v \notin \Gamma(v')$, for otherwise a dominates a' by the inequality (8). So,

$$\rho(v) < \max\{\rho(v'), \|vv'\|\}. \quad (9)$$

From the two inequalities the inequality (7) and (9), we have

$$\begin{aligned}&\max\{\rho(v), \rho(v')\}\\ &< \max\{\max\{\rho(v'), \|vv'\|\}, \max\{\rho(v), \|vv'\|\}\}\\ &= \max\{\rho(v), \rho(v'), \|vv'\|\},\end{aligned}$$

which implies that

$$\|vv'\| > \max\{\rho(v), \rho(v')\}. \quad (10)$$

The four inequalities (3), (5), (8) and (10) imply that the two links a and a' are independent, which is a contradiction. Therefore, the lemma holds. ■

Now, we are ready to describe the orientation H of G. For any pair of conflicting links a and a', we orient the edge in G between a and a' to an arc in H as follows: If they dominate each other, we take an arbitrary orientation; otherwise, if a dominates a', we take the orientation from a to a'; otherwise, we have that a' dominates a by Lemma 8 and take the orientation from a' to a. The lemma below establishes the upper bound 8 on the ILIN of H.

Lemma 9: The ILIN of H is at most 8.

The proof of Lemma 9 is quite geometrically involved, and is relegated to the Appendix. From Lemma 9, the third part of Lemma 6 follows immediately.

IV. Conclusion

All existing link scheduling in multihop wireless networks requires the positions, and/or communication/interference radii of all nodes. For practical networks, it is not only difficult or expensive to obtain these parameters, but also often impossible to get their precise values. The link schedules determined by the inaccurate values of these parameters may fail to guarantee the same approximation bounds with the link schedules determined by precise values. Therefore, the existing link scheduling algorithms lack of performance robustness. In this paper, we proposed a robust link scheduling, called first-fit fractional link scheduling in the smallest-CWD-last ordering. It can be easily computed with only the topological information on whether a given pair of links have conflict or not and the link demands. In addition, the approximation bound of this algorithm is no worse than the best-known approximation bound of existing link scheduling algorithms. Indeed, under the 802.11 interference model, its approximation bound is 16 in general and 6 with uniform interference radii, an improvement over the respective best-known approximation bounds 23 and 7 derived in [7]. Our general treatment of the first-fit fractional coloring in the smallest-CWD-last ordering is of independent interest and is expected to find other applications in the wireless scheduling problems.

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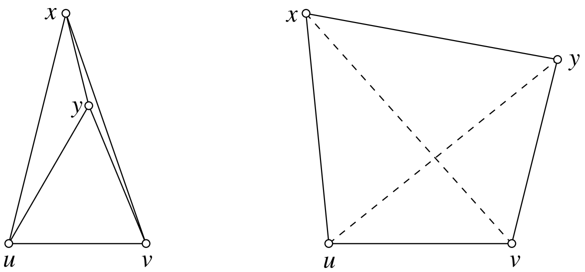
Appendix

In this appendix, we prove Lemma 9. We begin with two simple geometric facts about quadrilaterals. A quadrilateral is said to be *simple* if any pair of its non-adjacent sides do not cross each other.

Lemma 10: Consider a simple quadrilateral $uvyx$ satisfying that x and y lie on the same side of uv, $\|xy\| \geq \max\{\|xu\|, \|yv\|\}$, and $\|uv\| \leq \min\{\|xu\|, \|yv\|\}$. Then, the quadrilateral $uvyx$ is convex and $\angle xuv + \angle yvu \geq 180^o$.

Proof: We first prove that the quadrilateral $uvyx$ is convex. Assume to the contrary that quadrilateral $uvyx$ is not convex (see Figure 1(a)). Then, either y is inside $\triangle xuv$, or



(a) (b)

Fig. 1. Figure for Lemma 10.

x is inside $\triangle yuv$.By symmetry, we assume the former holds. Consider $\triangle yuv$. Since $\|yv\| \geq \|uv\|$, and hence $\angle uyv < 90^o$. Thus,

$$\angle xyu = 360^o - \angle uyv - \angle xyv > 360^o - 90^o - 180^o = 90^o.$$

So, $\|xu\| > \|xy\|$, which is a contradiction. Therefore, the quadrilateral $uvyx$ is convex.

Next, we prove that $\angle xuv + \angle yvu \geq 180^o$. We have four angle inequalities (see Figure 1(b)): $\angle uvx \geq \angle uxv$ since $\|xu\| \geq \|uv\|$; $\angle vuy \geq \angle uyv$ since $\|yv\| \geq \|uv\|$; $\angle xuy \geq \angle xyu$ since $\|xy\| \geq \|xu\|$; and $\angle yvx \geq \angle yxv$ since $\|xy\| \geq \|yv\|$. Summing up these four angle inequalities, we obtain

$$\angle xuv + \angle yvu \geq \angle uxy + \angle vyx.$$

Therefore, $\angle xuv + \angle yvu \geq 180^o$. ■

Lemma 11: Consider a convex quadrilateral $uvwy$ satisfying that

$$\|uy\| \geq \|wv\| = \|yw\| \geq \|uv\|$$

and $\angle uyv \geq 60^o$. Then, $\angle yuv + \angle wvu \geq 180^o$.

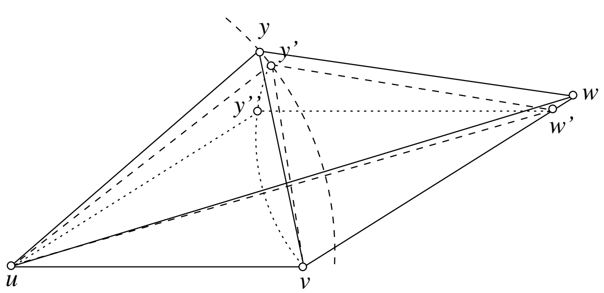


Fig. 2. Figure for Lemma 11.

Proof: Since $\|uy\| \geq \|uv\|$ and $\angle uyv \geq 60^o$, we have

$$\|uy\| \geq \|uv\| \geq \|vy\|.$$

Let w' be the point on vw such that $\|w'v\| = \|uv\|$ and y' be the point on the same side of uw' as w such that $\|y'u\| = \|yu\|$ and $\|y'w'\| = \|uv\|$ (see Figure 2). Then,

$$\|yw'\| \geq \|yw\| - \|w'w\| = \|vw\| - \|w'w\| = \|vw'\| = \|y'w'\|.$$

So, $y'u$ is between yu and $w'u$, and $\angle yuw' \geq \angle y'uw'$. Thus,

$$\angle yuv = \angle yuw' + \angle w'uv \geq \angle y'uw' + \angle w'uv = \angle y'uv.$$

Consequently,

$$\|y'v\| \leq \|yv\| \leq \|uv\| = \|w'v\|.$$

So,

$$\angle w'vy' = \angle w'y'v \geq 60^o,$$

and hence $\angle uvy' < 120^o$. Since $\|y'v\| \leq \|uv\|$, we have $\angle uy'v \geq \angle y'uv$ and hence

$$\angle uy'v \geq \frac{180^o - \angle uvy'}{2} \geq 30^o.$$

Thus,

$$\angle uy'w' = \angle uy'v + \angle vy'w' \geq 30^o + 60^o = 90^o.$$

Since $\|y'u\| \geq \|uv\|$, we have $\angle y'w'u \geq \angle vw'u$. Let y'' be such that $uvw'y''$ is a parallelogram (and hence is a rhombus). Then

$$\angle y''w'u = \angle vw'u \leq \angle y'w'u,$$

which means that $y''w'$ is between uw' and $y'w'$. As $\angle uy'w' \geq 90^o$, we have $\angle y'uw' \geq \angle y''uw'$, which implies $\angle y'uv \geq \angle y''uv$. Therefore,

$$\angle yuv \geq \angle y'uv \geq \angle y''uv,$$

and consequently

$$\angle yuv + \angle wvu \geq \angle y''uv + \angle wvu = 180^o.$$

■

We proceed to prove Lemma 9 using the above two lemmas. For each node v, we write B_v for the disk of radius $\rho(v)$ centered at v. A pair of nodes u and v are said to be *conflict-free* if

$$\|uv\| > \max\{\rho(u), \rho(v)\}.$$

Then, two links a and a' are conflict-free (or independent) if any endpoint of a and any endpoint of a' are conflict-free. We prove Lemma 9 by contradiction. Assume to the contrary that some link a has a set I of at least nine independent in-neighbors. Without loss of generality we assume that u and v are two endpoints of a with $\rho(u) \geq \rho(v)$ and u lying to the straight left of v. Note that all links in I are disjoint and none of them is the reverse of a. By the construction of H, each $a' \in I$ has an endpoint w satisfying that either $w \in \Gamma(u)$, or $w \in \Gamma(v)$ and $\|uw\| > \max\{\rho(w), \rho(u)\}$. We partition I into two subsets I_1 and I_2, where I_1 consists of all links $a' \in I$ which has an endpoint in $\Gamma(u)$, and I_2 consists of the rest links in I. For each link $a' \in I$, we choose an endpoint of a' as the representative of a' as follows. If $a' \in I_1$, its representative is an endpoint of a' which lies in $\Gamma(u)$; otherwise, its representative is an endpoint w of a' satisfying that $w \in \Gamma(v)$ and $\|wu\| > \max\{\rho(w), \rho(u)\}$. We denote the set of representative of the links in I_1 (respectively,

I_2) by S_1 (respectively, S_2). Let $S = S_1 \cup S_2$. Then, all nodes in S are conflict-free and

$$|S_1| + |S_2| = |S| = |I| \geq 9.$$

By the simple angle argument, the angle separation of any pair of distinct nodes in S_1 (respectively, S_2) is greater than 60^o. Therefore, $|S_1| \leq 5$. Since all nodes in S_2 lie on the right side of the perpendicular bisector of uv, we have $|S_2| \leq 4$. So,

$$|S_1| + |S_2| \leq 5 + 4 = 9.$$

Hence, $|S_1| = 5$ and $|S_2| = 4$. Let S_1' denote the set of nodes in S_1 which lie to the left side of the perpendicular bisector of uv and is outside B_v; and let $S_1'' = S_1 \setminus S_1'$. Then, $|S_1'| \leq 4$ and hence $|S_1''| \geq 1$. On the other hand, for each $w \in S_1''$,

$$\rho(w) \geq \max\{\rho(u), \|uw\|\} \geq \max\{\rho(v), \|vw\|\}.$$

Thus, $|S_1'' \cup S_2| \leq 5$ and hence $|S_1''| \leq 5 - |S_2| = 1$. Therefore, $|S_1'| = |S_2| = 4$ and $|S_1''| = 1$.

Let o be the upper intersection point of the perpendicular bisector of uv and ∂B_u. Number the nodes in S_1' by u_1, u_2, u_3 and u_4 such that o, u_1, u_2, u_3, u_4 are in the counterclockwise order with respect to u; and number the nodes in S_2 by v_1, v_2, v_3 and v_4 such that o, v_1, v_2, v_3, v_4 are in the clockwise order with respect to v (see Figure 3). Then, both u_2 and v_2 are strictly above the line uv, and both u_3 and v_3 are strictly below the line uv. Since

$$\begin{aligned} &\angle u_2uv + \angle u_3uv + \angle v_2vu + \angle v_3vu \\ &= 720^o - \angle u_2uu_3 - \angle v_2vv_3 \\ &< 720^o - 60^o - 60^o = 600^o, \end{aligned}$$

either $\angle u_2uv + \angle v_2vu$ or $\angle u_3uv + \angle v_3vu$ is less than 300^o. By symmetry, we assume that$\angle u_2uv + \angle v_2vu < 300^o$. We denote by B_v' the disk of radius $\rho(u)$ centered at v. Let $p \in \partial B_u$ be such that p is in the sector u_2uo and $\angle puu_2 = 60^o$, and $q \in \partial B_v'$ be such that q is in the sector v_2vo and $\angle qvv_2 = 60^o$. Then, u_1 lies inside the sector puo, v_1 lies inside the sector qvo, and

$$\angle puv + \angle qvu = \angle u_2uv + \angle v_2vu - 120^o < 180^o.$$

We shall show that

$$\|u_1v_1\| < \max\{\|u_1u\|, \|v_1v\|\} \leq \max\{\rho(u_1), \rho(v_1)\},$$

which is a contradiction.

If u_1 is outside B_u and v_1 is outside B_v, then by Lemma 10

$$\|u_1v_1\| < \max\{\|u_1u\|, \|v_1v\|\} \leq \max\{\rho(u_1), \rho(v_1)\},$$

which is a contradiction. So, we assume that either $u_1 \in B_u$, or $v_1 \in B_v$, or both. For each $1 \leq i \leq 4$, we denote the intersection of the ray uu_i (respectively, vv_i) and ∂B_u (respectively, $\partial B_v'$) by u_i' (respectively, v_i'). Let w be the intersection point of the line segment vv_2' and ∂B_v. We claim that if $u_1 \in B_u$, then $\|u_1u_2'\| > \rho(u)$. Assume to the contrary

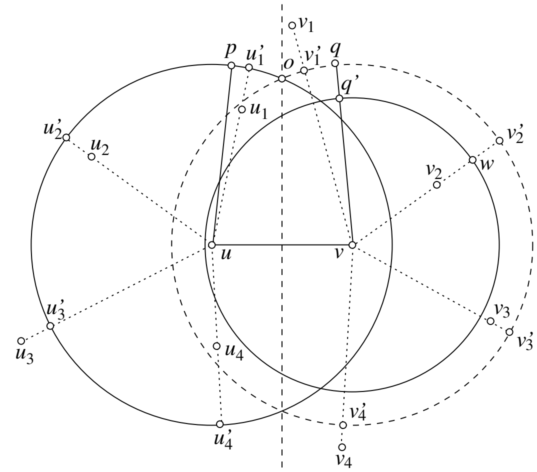


Fig. 3. The four nodes u_1, u_2, u_3 and u_4 in S_1' are sorted in the countercolockwise order with respect to u; and the nodes v_1, v_2, v_3 and v_4 in S_2 are sorted in the clockwise order with respect to v.

that $u_1 \in B_u$ and $\|u_1u_2'\| \leq \rho(u)$. If u_2 lies on the segment $u_2'u$, then

$$\|u_1u_2\| \leq \max\{\|u_1u_2'\|, \|u_1u\|\} \leq \rho(u),$$

which is a contradiction. So, we assume that u_2' is on the segment u_2u. Then,

$$\begin{aligned} \|u_1u_2\| &\leq \|u_2u_2'\| + \|u_1u_2'\| \leq \|u_2u_2'\| + \rho(u) \\ &= \|u_2u_2'\| + \|u_2'u\| = \|u_2u\| \leq \rho(u_2), \end{aligned}$$

which is also a contradiction. Therefore, our claim holds. Similarly, we can prove that if $v_1 \in B_v$, then $\|v_1w\| > \rho(v)$.

Let x be the intersection of ∂B_v and the circle of radius $\rho(u)$ centered at u_2' which lies on the same side of $u_2'v$ as p (see Figure 4). Since $\|pv\| > \|xv\|$, $u_2'x$ is between $u_2'v$ and $u_2'p$, and consequently, x is in the arc up of radius $\rho(u)$ centered at u_2'. So, $\|ux\| < \|up\|$ and hence x in inside B_u. We further show that $\angle qvx < 60^o$. Consider the quadrilateral $uvxu_2'$. Since

$$\|u_2'x\| = \|u_2'u\| \geq \|xv\| \geq \|uv\|,$$

we have $\angle u_2'uv + \angle xvu \geq 180^o$ by Lemma 10. So,

$$\begin{aligned} \angle qvx &= \angle qvu - \angle xvu < 180^o - \angle puv - \angle xvu \\ &= 180^o - (\angle u_2'uv - 60^o) - \angle xvu \\ &= 180^o - (\angle u_2'uv + \angle xvu) + 60^o \leq 60^o. \end{aligned}$$

We continue to derive contradictions in two cases.

Case 1: $q' \in B_u$ (see Figure 5). Let Ω be the region surrounded by the three line segments px, xq' and $q'q$ and the two arcs op and oq. We claim that the diameter of Ω is at most $\rho(u)$. Since $\angle qvx < 60^o$, the diameter of $\{x, q, q', q''\}$ is at most $\rho(u)$. Clearly, $\|px\| \leq \rho(u)$. By Lemma 10, $\|pq\| \leq \rho(u)$. Since $\|q'u\| \geq \|q'v\| \geq \|uv\|$, we have $\angle uq'v \leq 60^o$. Hence, $\angle puq' < \angle uq'v \leq 60^o$, which implies

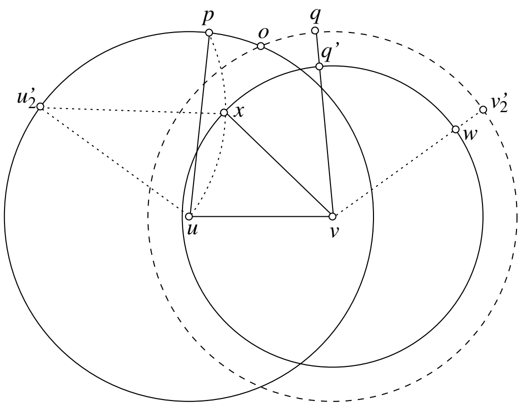


Fig. 4. $\angle qvx$ is less than 60^o.

that the diameter of $\{p, q', q''\}$ is at most $\rho(u)$. Thus, the diameter of $\{p, x, q, q', q''\}$ is at most $\rho(u)$. Since o is on the arc pq'', the diameter of $\{p, x, q, q', q'', o\}$ is at most $\rho(u)$. Hence, the diameter of Ω is at most $\rho(u)$. Since $q' \in B_u$ and $v_1 \notin B_u$, we have $v_1 \notin B_v$ and hence $u_1 \in B_u$. So, both u_1 and v'_1 lie in Ω, and hence $\|u_1v'_1\| \leq \rho(u)$. Thus,

$$\begin{aligned}\|u_1v_1\| &\leq \|u_1v'_1\| + \|v_1v'_1\| \leq \rho(u) + \|v_1v'_1\| \\ &= \|v'_1v\| + \|v_1v'_1\| = \|v_1v\| \leq \rho(v_1),\end{aligned}$$

which is a contradiction.

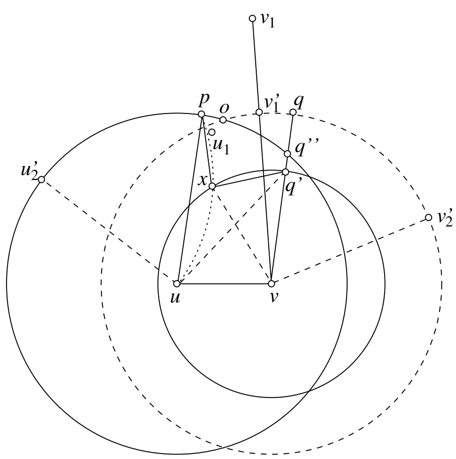


Fig. 5. When $q' \in B_u$, the diamater of $\{p, x, q, q', q'', o\}$ is at most $\rho(u)$.

Case 2: $q' \notin B_u$ (see Figure 6). Then,

$$\rho(v) \geq \|uv\| \geq \|uq'\| - \|q'v\| > \rho(u) - \rho(v),$$

and hence ∂B_u and ∂B_v cross each other. Let z be the upper intersection point of ∂B_u and ∂B_v. Then, z lies on the arc xq' of radius $\rho(v)$ centered at v, and consequently q' lies on the arc zw of radius $\rho(v)$ centered at v. Since $\|uw\| > \|uq'\| > \rho(u)$, $w \notin B_u$. Let y be the intersection of ∂B_u and the circle of radius $\rho(v)$ centered at w which lies on the same side of uw as o. Since $\|yu\| < \|q'u\|$, yw is between $q'w$ and uw, and consequently y is in the arc $q'v$ of radius $\rho(v)$ centered at w. So, $\|vy\| < \|vq'\|$ and hence y in inside B_v. So, vy is between vq and vz, and thus is also between vq and vx.

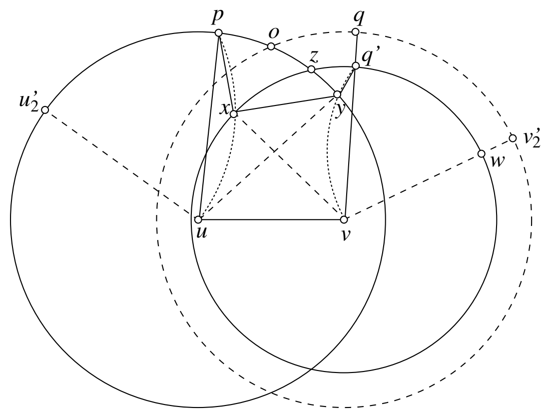


Fig. 6. When $q' \notin B_u$, the diamater of $\{o, p, q, q', x, y\}$ is at most $\rho(u)$.

Let Ω be the region surrounded by the four line segments px, xy, yq' and $q'q$ and the two arcs op and oq. We claim that the diameter of Ω is at most $\rho(u)$. Since $\angle qvx < 60^o$, the diameter of $\{q, q', x, y\}$ is at most $\rho(u)$. We show that $\angle puy < 60^o$. If $\angle uyv \leq 60^o$, then, $\angle puy < \angle uyv \leq 60^o$. So, we assume that $\angle uyv > 60^o$. By Lemma 11, $\angle yuv + \angle wvu \geq 180^o$. Thus,

$$\begin{aligned}\angle puy = \angle puv - \angle yuv &< 180^o - \angle qvu - \angle yuv \\ &= 180^o - (\angle wvu - 60^o) - \angle yuv \\ &= 180^o - (\angle yuv + \angle wvu) + 60^o \leq 60^o.\end{aligned}$$

Therefore, $\|py\| \leq \rho(u)$. By Lemma 10, both $\|pq\|$ and $\|pq'\|$ are at most $\rho(u)$. So, the diameter of $\{p, q, q', x, y\}$ is at most $\rho(u)$. Since o is on the arc py, the diameter of $\{o, p, q, q', x, y\}$ is at most $\rho(u)$. Hence, the diameter of Ω is at most $\rho(u)$. We consider the following three cases:

Case 2.1: $u_1 \in B_u$ and $v_1 \in B'_v$. Then both u_1 and v_1 lie in Ω and hence $\|u_1v_1\| \leq \rho(u) \leq \rho(u_1)$, which is a contradiction.

Case 2.2: $u_1 \notin B_u$ and $v_1 \in B'_v$. Then

$$\begin{aligned}\|u_1v_1\| &\leq \|u_1u'_1\| + \|u'_1v_1\| \leq \|u_1u'_1\| + \rho(u) \\ &= \|u_1u\| \leq \rho(u_1),\end{aligned}$$

which is a contradiction.

Case 2.3: $u_1 \in B_u$ and $v_1 \notin B'_v$. Then

$$\begin{aligned}\|u_1v_1\| &\leq \|u_1v'_1\| + \|v_1v'_1\| \leq \rho(u) + \|v_1v'_1\| \\ &= \|v'_1v\| + \|v_1v'_1\| = \|v_1v\| \leq \rho(v_1),\end{aligned}$$

which is a contradiction.

This completes the proof of Lemma 9.