Weakly-Connected Dominating Sets and Sparse Spanners in Wireless Ad Hoc Networks

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Abstract

A set S is dominating if each node in the graph $G = (V, E)$ is either in S or adjacent to at least one of the nodes in S. The subgraph weakly induced by S is the graph $G' = (V, E')$ such that each edge in E' has at least one end point in S. The set S is a weakly-connected dominating set (WCDS) of G if S is dominating and G' is connected. G' is a sparse spanner if it has linear edges. In this paper, we present two distributed algorithms for finding a WCDS in $O(n)$ time. The first algorithm has an approximation ratio of 5, and requires $O(n \log n)$ messages. The second algorithm has a larger approximation ratio, but it requires only $O(n)$ messages. The graph G' generated by the second algorithm forms a sparse spanner with a topological dilation *of* 3*, and a* geometric dilation *of* 6.

***Keywords**: weakly-connected dominating set, maximal independent set, sparse spanner.*

1. Introduction

Unlike wired networks or cellular networks, no physical backbone infrastructure is installed in wireless ad hoc networks. A communication session is achieved either through a single-hop radio transmission if the communication parties are close enough, or through relaying by intermediate nodes otherwise. In this paper, we assume that all nodes V in a wireless ad hoc network are distributed in a two-dimensional plane and have an equal maximum transmission range of one unit. The topology of a wireless ad hoc network can be modelled as a *unit-disk graph* (UDG) [10] $G = (V, E)$, a geometric graph in which there is an edge between two nodes if and only if their distance is at most one. (see Figure 1).

Although a wireless ad hoc network has no *physical* backbone infrastructure, a *virtual* backbone can be formed by nodes in a connected dominating set (CDS) of the corresponding unit-disk graph [6] [14]. In general, a *dominating set* (DS) of a graph $G = (V, E)$ is a subset $V' \subset V$ such that each node in $V - V'$ is adjacent to some node in V', and a CDS is a dominating set that also induces a connected subgraph. A (connected) dominating set of a wireless ad hoc network is a (connected) dominating set of the corresponding unit-disk graph. A virtual backbone, also referred to as a *spine*, plays a very important role in routing and broadcasting, where the number of nodes responsible for routing and broadcasting can be reduced to the number of nodes in the backbone [6]. To reduce the communication overhead, to increase the convergence speed, and to simplify the connectivity management, it is desirable to find a backbone of a small number of nodes. Several distributed algorithms have been proposed in the literature for constructing a CDS [14] [15][16], but all these algorithms suffer from a logarithmic or linear approximation ratio, and high time and message complexity [4]. Recently, several algorithms for constructed a CDS with constant approximation ratio, and linear time and messages were proposed in [2][3][4][5].

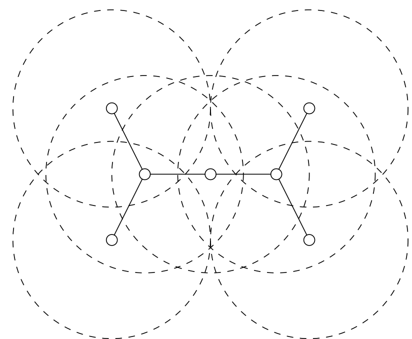

Figure 1. unit-disk graph

Another alternative for the CDS is the construction of

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a weakly-connected dominating set (WCDS) [8], a set that is dominating and all the edges with at least one end point in the set form a connected subgraph (*weakly induced* subgraph). However, it is NP-hard to find a minimum WCDS [11]. Approximation algorithms for a minimum WCDS have been proposed in the literature [8] with approximation ratio of $O(\ln \Delta)$, where Δ is the maximum nodal degree. The size of the minimum WCDS (MWCDS)is trivially smaller than or equal to the size of the MCDS, since we relax the connectivity requirement of the dominating set. Vertices 1 and 2 in Figure 2 are the weakly-connected dominating set, and the black edges show the weakly induced subgraph.

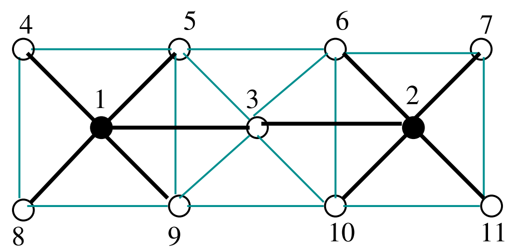

Figure 2. WCDS and its weakly induced graph

If the nodes are densely distributed, the unit-disk graph G may have $\Theta(n^2)$ edges where $n = |V|$. Running a networking operation such as all-pairs shortest path, routing and resource discovery directly over such dense unit-disk graph G often suffers from lack of scalability to potentially large network sizes. An effective approach to scalability is to run the networking operation over a *sparse spanner* $G' = (V, E')$ of the unit-disk graph G, i.e., a connected subgraph with $\Theta(n)$ edges [7] [12] [15]. A good quality WCDS for a graph G would have a *weakly induced* subgraph G' with linear edges (sparse spanner) and a constant dilation factor. The *topological dilation* factor for a spanner G' measures the worst-case ratio between the hops of a minimum-hop path in G' to the hops of a minimum-hop path between the same endpoints in G. But the *geometric dilation* of G' has to be defined as the worst-case ratio between the total length of a minimum- *hop* path in G' to the total length of a minimum-*distance* path in G between the same endpoints, since the minimum-distance path in G' is impossible to be obtained without knowing the positions of the networking nodes.

In this paper, we take advantage of the properties of the maximal independent set to construct weakly-connected dominating sets and their corresponding sparse spanners. A maximal independent set (MIS) for a graph $G = (V, E)$ is a subset V' of V, such that V' is an independent set (all pairwise nodes in V' are not adjacent) and no proper superset of V' is also independent. Consequently, any MIS is also a DS. We propose two algorithms for finding a WCDS S, and show that the subgraph weakly induced by S is a sparse spanner. The distributed construction of these spanners does not require knowledge about the geographic positions of the nodes at all. Each node is only required to know which nodes are in its vicinity, and consequently, the network operations can be more economically and reliably accomplished. Thus, the spanners generated by these algorithms are called position-less sparse spanners. The first distributed construction of a CDS-based position-less sparse spanner appeared in Alzoubi's PhD. thesis [1]. Our first algorithm produces a sparse spanner based on a WCDS with low constant ratio of 5. The second algorithm generates a sparse spanner with small constant *topological dilation* and *geometric dilation*.

The paper is organized as follows. In Section 2, we study the properties of the MIS. In Section 3, we discuss the relation between the topological dilation and the geometric dilation. In Section 4, we describe new algorithms for the construction of a WCDS and its induced spanner. Finally, we conclude this paper in section 5.

2. Maximal independent set

A set S is an independent set, if all pairwise nodes in S are not adjacent, i.e. there is no edge between any pair of nodes in S. An independent set S is maximal if no proper superset of S is also an independent set [13]. This definition implies that for any graph, if a node is not in the maximal independent set (MIS), then it must be adjacent to at least one of the nodes in the MIS. Thus, the MIS of any graph is an independent dominating set (IDS). The domination property of the MIS, and the sparseness of its nodes provides a motivation to study its other properties. In this section we investigate other properties of the MIS, and present a new approach for the construction of the MIS [3]. This approach proposes a new ranking method, where the rank is a unique identifier of the node, and the ranks of all the nodes can be sorted in ascending, or descending order.

2.1. MIS properties

This section introduces important properties that contribute to the development of an efficient virtual backbone for wireless ad hoc networks.

Lemma 1 *Let G be a unit-disk graph, and let S be the MIS, then any node in G and not in S has at most five neighbors in S.*

In Figure 3, let u be the node not in S, and let the black nodes be the nodes in S. It is obvious from the figure that

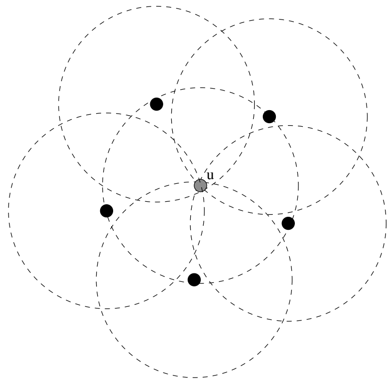

Figure 3. Any node has at most 5 MIS nodes

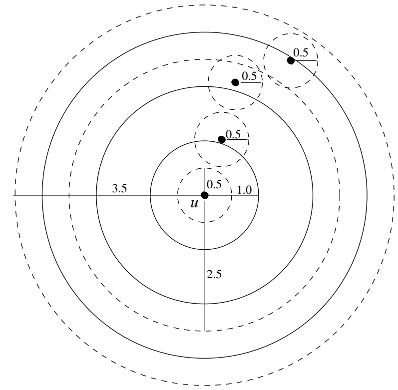

Figure 4. Dominators in 3-unit radius

any node not in S is adjacent to at most five nodes in S. Otherwise, at least two of the nodes from S must have an edge, which violates the definition of the MIS. Detailed proof is available in [13].

The next lemma provides a bound on the number of MIS nodes within two-hop or three-hop distance from any MIS node.

Lemma 2 *Let S be any MIS of the unit-disk graph G and u be an arbitrary node in S.*

1. *The number of nodes in S that are exactly two hops away from u is at most 23.*
2. *The number of nodes in S that are at most three hops away from u is at most 47.*

Proof. The proof follows from the standard area argument. The disks of radius 0.5 centered at the nodes in S that are exactly two hops away from u all lie within the annulus centered at u of radii 0.5 and 2.5 and are disjoint (see Figure 4). Thus, the number of nodes in S that are exactly two hops away from u is less than

$$\frac{\pi \cdot 2.5^2 - \pi \cdot 0.5^2}{\pi \cdot 0.5^2} = 24.$$

Furthermore, the disks of radius 0.5 centered at nodes in S that are at most three hops away from u all lie within the annulus centered at u of radii 0.5 and 3.5 and are disjoint (see Figure 4). So the number of nodes in S that are exactly three hops away from u is less than

$$\frac{\pi \cdot 3.5^2 - \pi \cdot 0.5^2}{\pi \cdot 0.5^2} = 48.$$

This completes the proof. ■

Lemma 3 *Let G be a unit-disk graph, and let S be the MIS, then the shortest-hop path between any two complementary subsets A and B of S is either two-hop or $three$-hop distance.*

Proof. It is obvious that a $single$-hop path is not possible, since the MIS nodes can't be adjacent. Let A be a subset of the MIS S, and let B be the complementary subset of A (see Figure 5). Let us assume the $four$-hop path (a, u, v, w, b) is the shortest-hop path between the two subsets A and B, where $a \epsilon A$ and $b \epsilon B$. Since the node v must be adjacent to at least one of the nodes in A or B, or in both, then there must be a $three$-hop or two-hop path between the two complementary subsets A and B, which contradicts our assumption. ■

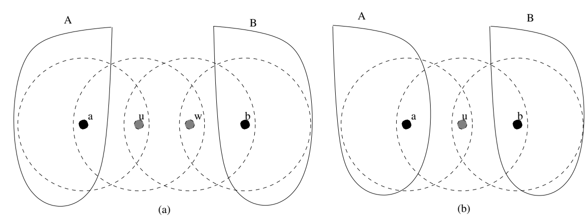

Figure 5. Complementary subsets of MIS

2.2. MIS and ranking

The centralized construction of the MIS in a unit-disk graph can be done in the following simple way: Initially all nodes are unmarked (white). While there are some unmarked nodes, select an arbitrary unmarked node v, mark it black and mark all its neighbors gray. When all nodes are marked, all black nodes form a MIS. However, this straightforward construction produces an arbitrary MIS with the property described in Lemma 3.

In some cases, the induced MIS needs to maintain some predefined properties. Thus, additional strict restrictions are imposed on the selection of the MIS nodes. It is fundamental for the nodes in the graph to be identified uniquely by assigning a unique rank for each node [3]. This process of identification is called ranking. The importance of ranking appears during the construction process to break ties in case of symmetry .

Ranking can be divided into two types, static ranking, where the rank of a node does not change during the construction process; one example is the node's ID. The second type is dynamic ranking, where the rank of the node may change during the construction process, examples, the ordered pair (degree, ID) or (degree, location). During the construction process of the MIS, the degree of a node could be the number of adjacent white nodes. The ID or the location are used as a second criteria to break ties in case of symmetry.

In this section, and for completeness purposes, we provide a detailed discussion of the ranking approach in [3]. The rank of the node in this approach consists of the ordered pair (level, ID). To assign a level for each node, first we build an arbitrary spanning tree T rooted at a node v. After such construction is completed, then the level of each node is identified by its hop-distance from the root of the spanning tree (i.e., its graph distance in T from the root) (see Figure 6). The ranks of all nodes are sorted in the lexicographic order. Thus the root, which is at level zero, has the lowest rank. The node with ID 10 at level one has the rank (1, 10). Similarly the node with ID seven at level three has the rank (3, 7). This type of ranking is called level-based ranking. The goal of this ranking approach is to construct a MIS with the property: for any two complementary subsets of the MIS, the shortest hop-path between the two subsets is exactly two hops. The MIS of this property is also a WCDS, follow the proof in Theorem 5.

The MIS induced by the level-based ranking can be summarized by the following: let U be the MIS, and initially is empty, and let V be the set of all nodes in the graph. The algorithm is described in Table 1.

The next theorem, shows that the MIS constructed by using the Level-Based approach for rank assignment guarantees that the distance between any pair of complementary subsets is *exactly* two hops.

Theorem 4 *The distance between any two complementary subsets of the MIS U constructed by the above algorithm is exactly two hops.*

Proof. Let $U = \{u_i : 1 \leq i \leq k\}$ where u_i is the i^{th} node that is marked black. For any $1 \leq j \leq k$, let H_j be the graph over $\{u_i : 1 \leq i \leq j\}$ in which a pair of nodes are connected by an edge if and only if their graph distance in G is exactly two hops. We prove by induction on j that H_j is connected. Since H_1 consists of a single vertex, it is connected trivially. Assume that H_{j-1} is connected for some $j \geq 2$. When the node u_j is marked black, non of its neighbors can be marked black, and its parent in T must be already marked gray. Thus, there is a black node u_i with $1 \leq i < j$ which is adjacent to u_j's parent in T. So u_i and u_j are separated by two hops, and thus (u_i, u_j) is an edge in H_j. As H_{j-1} is connected, so H_j must be also. Therefore, H_j is connected for any $1 \leq j \leq k$. The connectedness of H_k then implies that any pair of complementary subsets of U are separated by exactly two hops. ■

Table 1. MIS construction

- $U = \emptyset$
- While ($V \neq \emptyset$)
  - Let w be the node in V with the lowest rank
  - Add w to U
  - Mark w black, and all its neighbors gray
  - Remove w and all its neighbors from V
- Return U

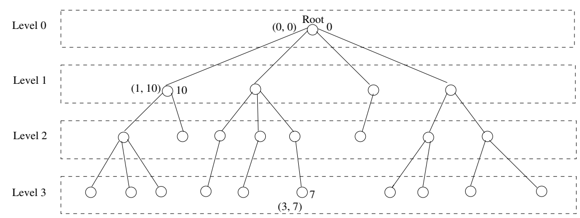

Figure 6. Level-based ranking

The next theorem, shows that the MIS constructed by using the Level-Based approach for rank assignment is also a WCDS.

Theorem 5 *Any MIS S constructed by using the Level-Based approach for rank assignment is also a WCDS.*

Proof. To prove that S is a WCDS, we need to prove that S is a dominating set, and all the edges incident on at least one MIS node form a connected subgraph. Since S is an MIS then it is a dominating set by definition. Let us color all edges that are incident at MIS nodes with black, and color the rest of the edges with white. Thus, the weakly induced subgraph G' consists of all black edges. We prove by contradiction that G' is connected. Let us assume that G' is not connected, then there must be at least two complementary subsets of the MIS nodes, that are separated by

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at least one white edge, and no black-edge path exists between them. By theorem 4 the shortest hop path between the two complementary subsets is exactly two hops. Since there is exactly one intermediate node that is not an MIS node between the two subsets, and this node is adjacent to at least one MIS node in each subset, then both edges must be black. Thus, there is a black-edge path between the two subsets, and G' is connected. ■

3. Relation between topological dilation and geometric dilation

Let G' be any spanner of G. Let u and v be any pair of nodes in V. We use $||uv||$ to denote the Euclidean distance between u and v. The minimum number of hops in G (respectively, G') between u and v is denoted by $h_G(u,v)$ (respectively, $h_{G'}(u,v)$). The total length of the minimum-distance path in G between u and v is denoted by $l_G(u,v)$. The maximum total length of the minimum-hop paths in G' between u and v is denoted by $l_{G'}(u,v)$. Note that $||uv|| \leq l_G(u,v)$, and the equality holds if $l_G(u,v) \leq 1$.

Lemma 6 *Let $u, v \in V$ be any pair of non-adjacent nodes in G, and let α and β be two constant numbers. If*

$$h_{G'}(u,v) \leq \alpha h_G(u,v) + \beta,$$

then

$$\ell_{G'}(u,v) < 2\alpha \ell_G(u,v) + \alpha + \beta.$$

Proof. Let $u = v_0, v_1, \cdots, v_k = v$ be a minimum-distance path in G between u and v with smallest number of hops k among all those minimum-distance paths in G between u and v. Note that $k \geq 2$ as u and v are not adjacent. We first claim that for any $0 \leq i \leq k-1$,

$$\ell_G(v_i, v_{i+1}) + \ell_G(v_{i+1}, v_{i+2}) > 1.$$

Assume to the contrary that for some $0 \leq i < k-1$,

$$\ell_G(v_i, v_{i+1}) + \ell_G(v_{i+1}, v_{i+2}) \leq 1.$$

Then

$$\ell_G(v_i, v_{i+2}) \leq \ell_G(v_i, v_{i+1}) + \ell_G(v_{i+1}, v_{i+2}) \leq 1.$$

This implies that v_i and v_{i+2} are also adjacent in G. So the path $v_0, v_1, \cdots, v_i, v_{i+2}, \cdots, v_k$ is also a minimum-distance path in G between u and v. But its number of hops is $k-1$, a contradiction to the definition of k. Therefore, our claim is true. Consequently,

$$\begin{aligned}\ell_G(u,v) &= \sum_{i=0}^{k-1} \ell_G(v_i, v_{i+1}) \\ &\geq \sum_{i=0}^{2\lfloor \frac{k}{2} \rfloor - 1} \ell_G(v_i, v_{i+1}) \\ &> \left\lfloor \frac{k}{2} \right\rfloor.\end{aligned}$$

Now assume that

$$h_{G'}(u,v) \leq \alpha h_G(u,v) + \beta.$$

Since

$$h_G(u,v) \leq k,$$

we have

$$h_{G'}(u,v) \leq \alpha h_G(u,v) + \beta \leq \alpha k + \beta.$$

Therefore,

$$\begin{aligned}\ell_{G'}(u,v) &\leq h_{G'}(u,v) \leq \alpha k + \beta \\ &\leq \alpha \left(2 \left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \beta \\ &= 2\alpha \left\lfloor \frac{k}{2} \right\rfloor + \alpha + \beta \\ &< 2\alpha \ell_G(u,v) + \alpha + \beta.\end{aligned}$$

This completes the proof. ■

4. WCDS and its induced spanner

In this section, we propose two algorithms to construct the WCDS and its induced sparse spanner. The first algorithm constructs an MIS using the level-based approach for ranking [2], then based on Theorem 5 the MIS is the WCDS. and all the edges incident at the MIS nodes are colored black, and form the sparse spanner. The WCDS U in the second algorithm is a union of two subsets of nodes, the MIS S (MIS-dominators) and subset C (additional-dominators). The MIS in this algorithm does not require the spanning tree, and the rank of each node is only the node's ID. Thus, the shortest-hop path between any two complementary subsets of this MIS is either two-hop or $three$-hop path. For each pair of MIS-dominators that are exactly $three$-hop apart, one intermediate node is selected as an additional-dominator and added to the set C. After the completion of C, all the nodes in U form the WCDS and all the edges incident at the nodes in U are colored black and form the sparse spanner.

4.1. Algorithm I

By Theorem 5 the construction of an MIS using the level-based approach for ranking is a construction of a WCDS. Our distributed algorithm for WCDS consists of three phases: the Leader Election Phase, the Level Calculation Phase, and the Color Marking Phase. The Leader Election Phase elects a leader v and constructs a spanning tree T rooted at the leader. The distributed algorithm in [9] for leader election can be adopted. Note that any criteria can be used to define the leadership, such as ID or the combination of degree and ID. This algorithm has $O(n)$ time complexity and $O(n \log n)$ message complexity. At the end of the first phase, each node knows its parent and its children in T.

In the Level Calculation Phase, each node identifies its level in T. It starts with the root announcing its level 0. Each node, upon receiving the level announcement message from its parent in T, obtains its own level by increasing the level of its parent by one, and then announces this level. Each node also records the levels of its neighbors in the unit-disk graph. If we need to report the completion of the tree, a report process has to be performed upwards along the T. When a leaf node has determined its level, it transmits a COMPLETE message to its parent. Each internal node will wait till it receives this COMPLETE message from each of its children and then forward it up the tree toward the root. When the root receives the COMPLETE message from all its children, then it starts the third phase. Obviously, the total number of messages sent in this phase is $O(n)$. At this moment, each node knows the levels and IDs of its own and its neighbors. The pair (level, ID) of a node defines the *rank* of this node. The ranks of all nodes are sorted in the lexicographic order. Thus the leader, which is at level 0, has the lowest rank.

In the Color Marking Phase, all nodes are initially unmarked (white), and will eventually get marked either black or gray. Two types of messages are used by the nodes during this phase, the BLACK message and the GRAY message. The BLACK message is sent by a node after it marks itself black, and the GRAY message is sent by a node after it marks itself gray. Both messages contains the sender's ID. This phase is initiated by the root which marks itself black, and then broadcasts to its neighbors a BLACK message. All other nodes act according to the following principles.

- Whenever a white node receives a BLACK message for the *first* time, it marks itself gray and broadcasts a GRAY message.
- After a white node has received a GRAY message from *all* of its neighbors of lower rank, it marks itself black and broadcasts a BLACK message.
- All edges incident at the black nodes are colored black.

The time and message complexity of this algorithm is dominated by the complexity of the leader election process, which is $O(n)$ for the time complexity, and $O(n \log n)$ for the message complexity. After the leader is found, only linear number of messages is needed as each node sends only a constant number of messages.

The next lemma shows that the size of the above WCDS is at most five times the size of the MWCDS.

Lemma 7 *Let opt be the size of a MWCDS for the unit-disk graph $G = (V, E)$, then the size of the WCDS generated by Algorithm I for the graph G is at most $5 \cdot$ opt.*

Proof. By definition, each node in the MIS is either in the MWCDS, or it must be adjacent to at least one of the nodes in the MWCDS, and the neighborhood of each node in the MWCDS contains at most $five$ vertices from the MIS. Thus, the size of the MIS is $\leq 5 \cdot opt$. Since the WCDS generated by Algorithm I is a MIS, then $|WCDS| \leq 5 \cdot opt$. This completes the proof. ■

Theorem 8 *The subgraph of the black edges generated by the Algorithm I forms a sparse spanner.*

Proof. Since each gray node has an edge with at least one black node and by Theorem 5 the subgraph is connected, then the subgraph is a spanner. By construction, each black edge is an edge between a gray node and a black node. If we charge all black edges to the gray nodes, and since each gray node has at most $five$ black nodes, then the maximum number of black edges is $five$ times the number of gray nodes. Thus, the spanner is sparse. ■

Lemma 9 *If the shortest-hop path between any two complementary subsets of any dominating set S is at most two hops, then S is a WCDS.*

Proof. Based on Theorem 5, when the shortest hop path between any two complementary subsets is two hops, then S is a WCDS. When the shortest hop distance between any two complementary subsets is *one* hop, then the edge connecting the two subsets is a black edge, and thus the weakly induced subgraph is connected, i.e. S is a WCDS. ■

4.2. Algorithm II

The WCDS U constructed by this algorithm consists of two sets of nodes, MIS-dominators set and additional-dominators set. Based on Lemma 9, if the shortest hop path between any two complementary subsets of a DS S is at most two hops, then S is a WCDS. This algorithm takes advantage of this lemma, and constructs a WCDS by constructing a DS with the property specified in Lemma 9. The construction of such a DS consists of three phases: the first phase we construct an arbitrary MIS, such an MIS may have the property specified in Lemma 3. Thus, it does not guarantee the exactly *two* hop distance between an arbitrary two complementary subsets. The nodes of the MIS are called MIS-dominators. The second phase modifies the MIS constructed in the first phase to a DS with the property specified in Lemma 9. This is done by selecting one intermediate node between each pair of MIS-dominators separated by exactly three hops. These selected nodes are called additional-dominators. Both of the MIS-dominators and the additional-dominators form the WCDS. In the third phase, each edge incident at any of the dominators is colored black. The subgraph with all black edges forms the sparse spanner.

The algorithm can be described as follow: we assume that each node has a unique ID, and knows the IDs of all its neighboring nodes. Each node is in one of the three colors: white, gray, or black. Each node initially has the color white, and subsequently is colored either black or gray. Each gray node maintains two lists: 1HopDomList, which contains the IDs of all dominators that are 1-hop away from itself. And 2HopDomList, where each entry in the list consists of the ID of the 2-hop dominator, and the ID of an adjacent node to reach this dominator. Each MIS-dominator maintains two lists: the 2HopDomList, and a 3HopDomList, where each entry in the list contains the ID of a 3-hop dominator, and the IDs of the two intermediate nodes to reach this dominator. The algorithm is summarized by the following steps.

Each node which has the lowest ID among all its white neighbors, colors itself black and declares itself as a dominator by sending a MIS-DOMINATOR message to all its neighbors. Note that such node does exist.

Whenever a white node receives a MIS-DOMINATOR message for the *first* time, it colors itself gray, adds the ID of the sender to its 1HopDomList, and sends a GRAY message.

Whenever a white node has received the GRAY messages from all of its neighbors with lower IDs, if there is any, it colors itself black and declares itself as a dominator by broadcasting a MIS-DOMINATOR message.

Whenever a gray node has received GRAY or MIS-DOMINATOR messages from all its neighbors, it sends a 1-HOP-DOMINATORS message, which contains its own ID, and its own 1HopDomList.

Whenever a gray node receives a 1-HOP-DOMINATORS message from a neighbor, for each dominator ID in the received message and not in its 1HopDomList it adds to its 2HopDomList the ID of the 2-hop dominator, and the ID of the sender, which is the intermediate node to reach the 2-hop dominator.

Whenever a MIS-dominator receives a 1-HOP-DOMINATORS message from a neighbor, for each dominator ID in the received message different from its own ID and not in its 2HopDomList it adds to its 2HopDomList the ID of the 2-hop dominator, and the ID of the sender, which is the intermediate node to reach the 2-hop dominator. If any dominators ID exists in its 3HopDomList, it removes the corresponding entry from its 3HopDomList.

Whenever a gray node has received 1-HOP-DOMINATORS message from each gray neighbor, it sends 2-HOP-DOMINATORS message, which contains its own ID and its own 2HopDomList.

Whenever a MIS-dominator u receives a 2-HOP-DOMINATORS message from a neighbor v, for each entry (w, x) in the 2HopDomList of the received message if the dominator w is not in u's 2HopDomList or in u's 3HopDomList, and the ID of u is smaller than the ID of w it adds an entry of (w, v, x) to its 3HopDomList. Where w is the 3-hop dominator, and the nodes v and x are the intermediate nodes to reach $w$message to v, which contains the nodes (u, v, x, w).

Whenever a node v receives the SELECTION message addressed to itself, it colors itself black and declares itself as an additional-dominator by broadcasting an ADDITIONAL-DOMINATOR message, which contains the nodes (v, u, x, w).

Whenever the MIS-dominator w receives the ADDITIONAL-DOMINATOR message, it adds the entry (u, x, v) to its 3HopDomList. Where u is the 3-hop dominator, and the nodes x and v are the intermediate nodes to reach w.

Both of the MIS-dominators and the additional-dominators form the WCDS U. Each edge incident at one or two nodes from U is colored black. The subgraph G' which contains all black edges is the weakly induced subgraph, and it is a sparse spanner with constant dilation factor.

For any pair of adjacent nodes in G, the unicast routing between them can be performed in a single hop. For any pair of non-adjacent nodes in G, the unicast routing between them will follow the min-hop path in the spanner G'. The MIS-dominators (clusterhead) maintain the routing tables. If a non MIS-dominator node needs to send a packet to a non-adjacent node, it sends the packet along with the destination's ID to its clusterhead. The clusterhead uses its routing tables to identify the next clusterhead on the path to the destination's clusterhead, and uses its 2HopDomList and 3HopDomList to identify the path to the next clusterhead.

The WCDS obtained by this algorithm is easy to maintain whenever the nodes move around or are turned off or on. In the mean time we should be able to maintain the same performance ratio, and the same properties of the original constructed WCDS. The key technique in our approach is to maintain the MIS in the unit-disk graph at all time, and to maintain information about all MIS-dominators within

$three$-hop distance. The MIS-dominator u with a lower ID than its $three$-hop distance MIS-dominator v, maintains an additional dominator w, to be able to reach the node v through a black-edges path. Notice, the algorithm can be applied locally, and the nodes that get affected are within $three$-hop distance. During the maintenance process of the WCDS, we need to distinguish between MIS-dominators and additional-dominators. A detailed description of the maintenance process will appear in a different paper.

In the following theorem we prove that the WCDS generated by Algorithm II has a constant approximation ratio, and G' is a sparse spanner.

Theorem 10 *Given a unit-disk graph $G(V, E)$, the size of the WCDS U generated by Algorithm II is within a constant factor of opt, and the weakly induced subgraph $G'(V, E')$ is a sparse spanner.*

Proof. From Lemma 2, the total number of pairs of nodes in the MIS S that are within three hops away from each other is at most $\frac{47|S|}{2}$. Since each of such pairs introduces at most one node to the additional-dominator set C, then the total number of nodes introduced to C is at most $\frac{47|S|}{2}$. From Lemma 7, the total number of nodes in the WCDS U is at most

$$\frac{47\,|S|}{2} + |S| = \frac{47 \cdot (5 \cdot opt)}{2} + 5 \cdot opt = 122.5 \cdot opt$$

Thus, the size of U is within a constant factor of opt.

We have three types of edges: 1) An edge between a gray node and a node from S. Since each gray node is adjacent to at most 5 nodes from S, then we have at most $5 \cdot |G|$ edges, where $|G|$ is the number of gray nodes. 2) An edge between a node from S and a node from C, and we have at most $\frac{47|S|}{2}$ edges. 3) An edge between a gray node and a node from C. Each node in C must be adjacent to a node from S. Since each gray node has at most 24 nodes from S within 2-hop distance, and each one of these nodes introduces at most one node to C, then each gray node is adjacent to at most 24 nodes from C, and we have at most $24 \cdot |G|$ of this type of edges. Thus, the total number of edges is $29 \cdot |G| + \frac{47|S|}{2} = \Theta(|V|)$, and G' is a sparse spanner. ■

The bound on the size of U may be improved by tighter analysis.

The following theorem implies the constant topological dilation and constant geometric dilation of G'.

Theorem 11 *Let $u, v \in V$ be any pair of non-adjacent nodes in G. Then*

$$h_{G'}(u, v) \leq 3h_G(u, v) + 2;$$
$$\ell_{G'}(u, v) \leq 6\ell_G(u, v) + 5.$$

Proof. Let $u = v_0, v_1, \cdots, v_k = v$ be a minimum-hop path in G between u and v, where $k = h_G(u, v) \geq 2$. For any $0 \leq i \leq k$, let u_i be v_i itself if $v_i \in S$; otherwise, let u_i be any node in S that is a neighbor of v_i in G. Then for any $0 \leq i < k$,

$$h_G(u_i, u_{i+1}) \leq 3.$$

From the selection of C, we have

$$h_{G'}(u_i, u_{i+1}) = h_G(u_i, u_{i+1}).$$

Therefore,

$$h_{G'}(u_i, u_{i+1}) \leq 3.$$

This implies that

$$h_{G'}(u_0, u_k) \leq \sum_{i=0}^{k-1} h_{G'}(u_i, u_{i+1}) \leq 3k.$$

Thus,

$$\begin{aligned} h_{G'}(u, v) &= h_{G'}(v_0, v_k) \\ &\leq h_{G'}(v_0, u_0) + h_{G'}(u_0, u_k) + h_{G'}(u_k, v_k) \\ &\leq 1 + 3k + 1 = 3k + 2 \\ &= 3h_G(u, v) + 2. \end{aligned}$$

From Lemma 6, we have

$$\ell_{G'}(u, v) \leq 6\ell_G(u, v) + 5.$$

This completes the proof. ■

Theorem 12 *Our distributed algorithm for constructing a WCDS has an $O(n)$ time complexity, and $O(n)$ message complexity.*

Proof. The worst case time complexity for the MIS occurs when all nodes are arranged in either ascending or descending order and the maximum nodal degree is 2. In this case each node has to wait for all other nodes with lower ids. Assume we have the graph with the n nodes $(v_1, v_2, ..., v_n)$, then each node v_i must wait for its neighbor node v_{i-1} to declare its state. Each node must wait one time unit more than the waiting time of the previous node. Node v_n has to wait the longest ($n - 1$ units). Also each node sends only one message either a MIS-DOMINATOR or GRAY message. To find the additional-dominators, each gray node waits $O(\Delta)$ time to build its 1HopDomList and 2HopDomList. A MIS-dominator node waits $O(\Delta)$ time for 1-HOP-DOMINATORS and 2-HOP-DOMINATORS messages from all its neighbors before it selects an additional-dominator. The time required to perform the rest of the procedures is constant. Since each node sends a constant number of messages, the total number of messages is $O(n)$. Thus, each of the time complexity and the message complexity of our algorithm is $O(n)$. ■

5. Conclusion

This paper describes important properties of the maximal independent set, and the importance of these properties for constructing a weakly connected dominating set of small size. Two algorithms for constructing WCDS based on the MIS were introduced. The first algorithm produces a WCDS with approximation ratio of 5, and the weakly induced subgraph is a sparse spanner. The time and message complexity of this algorithm is $O(n)$ and $O(n \log n)$ respectively. Nodes are only required to maintain information about *single*-hop neighbors. The second algorithm has a higher approximation ratio of 122.5, but this algorithm has an optimal time and message complexity of $O(n)$, furthermore, the induced sparse spanner of this algorithm has a topological dilation of 3, and geometric dilation of 6. This algorithm is fully localized, which makes it practical for mobility environment. Both of these algorithms do not require geographic location information of the nodes.

References

[1] K. M. Alzoubi, "Virtual Backbones in Wireless Ad Hoc Networks", PhD dissertation, Department of Computer Science, Illinois Institute of Technology, Chicago, May 2002.

[2] K. M. Alzoubi, P.-J. Wan, O. Frieder, "Distributed Construction of Connected Dominating Set in Wireless Ad Hoc Networks", to appear in *Special Issue of ACM Journal of MONET*, 2002.

[3] K. M. Alzoubi, P.-J. Wan, O. Frieder, "Distributed Heuristics for Connected Dominating Set in Wireless Ad Hoc Networks", *IEEE ComSoc/KICS Journal of Communications and Networks*, vol. 4(1), pp. 22-29. March 2002.

[4] K. M. Alzoubi, P.-J. Wan, O. Frieder, "New Distributed Algorithm for Connected Dominating Set in Wireless Ad Hoc Networks", *IEEE HICSS35, Hawaii*, January 2002.

[5] K. M. Alzoubi, P.-J. Wan, O. Frieder, "Message-Optimal Connected-Dominating-Set Construction for Routing in Mobile Ad Hoc Networks", *MobiHoc 2002, the Third ACM International Symposium on Mobile Ad Hoc Networking and Computing*, June 2002.

[6] V. Bharghavan and B. Das, "Routing in Ad Hoc Networks Using Minimum Connected Dominating Sets", *International Conference on Communications'97*, Montreal, Canada. June 1997.

[7] P. Bose, P. Morin, I. Stojmenovic, and J. Urrutia,"Routing with guaranteed delivery in ad hoc wireless networks", in *3rd Int. Workshop on Discrete Algorithms and methods for mobile computing and communications*, 1999..

[8] Y. P. Chen, A. L. Liestman, "Approximating Minimum Size Weakly-Connected Dominating Sets for Clustering Mobile Ad Hoc Networks", *MobiHoc 2002, the Third ACM International Symposium on Mobile Ad Hoc Networking and Computing*, June 2002.

[9] I. Cidon and O. Mokryn, "Propagation and Leader Election in Multihop Broadcast Environment", *12th International Symposium on Distributed Computing (DISC98)*, September 1998, Greece. pp.104–119.

[10] B. N. Clark, C. J. Colbourn, and D. S. Johnson, "Unit Disk Graphs", *Discrete Mathematics*, 86:165–177, 1990.

[11] J. E. Dunbar, J. W. Grossman, J. H. Hattingh, S. T. Hedetniemi, and A. A. McRae. "On Weakly-connected Domination in Graphs", *Discrete Math*, 167/168: 261-269, 1997.

[12] B. Karp and H. T. Kung, "GPSR: Greedy perimeter stateless routing for wireless networks", in *ACM/IEEE International Conference on Mobile Computing and Networking*, 2000.

[13] M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi and D. J. Rosenkrantz, "Simple Heuristics for Unit Disk Graphs", *Networks*, Vol. 25, 1995, pp. 59–68.

[14] R. Sivakumar, B. Das, and V. Bharghavan, "An Improved Spine-based Infrastructure for Routing in Ad Hoc Networks", *IEEE Symposium on Computers and Communications '98*, Athens, Greece. June 1998.

[15] M. Seddigh, J. Solano and I. Stojmenovic, "RNG and internal node based broadcasting in one-to-one wireless networks", *ACM Mobile Computing and Communications Review*, Vol. 5, No. 2, April 2001, 37-44.

[16] J. Wu and H.L. Li, "On calculating connected dominating set for efficient routing in ad hoc wireless networks", *Proceedings of the 3rd ACM international workshop on Discrete algorithms and methods for mobile computing and communications*, 1999, Pages 7–14.

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