Imperative Languages and State

$$
\begin{aligned}
e &::= x \mid \bar{n} \mid \mathsf{true} \mid \mathsf{false} \mid e \overset{+,-,\dots}{\oplus} e \mid e < e \\
s &::= x \leftarrow e \mid s ; s \mid \mathsf{while}\ e\ s \mid \mathsf{if}\ e\ s_1\ \mathsf{else}\ s_2 \mid \mathsf{skip} \\
\tau &::= \mathsf{int} \mid \mathsf{bool}
\end{aligned}
$$

$\Gamma \vdash e : \tau$ $\qquad \dfrac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \dfrac{\Gamma \vdash e_1 : \mathsf{int} \quad \Gamma \vdash e_2 : \mathsf{int}}{\Gamma \vdash e_1 \oplus e_2 : \mathsf{int}}$

$$\dfrac{\Gamma \vdash e_1 : \mathsf{int} \quad \Gamma \vdash e_2 : \mathsf{int}}{\Gamma \vdash e_1 < e_2 : \mathsf{bool}} \qquad \dots$$

$\Gamma \vdash s\ \mathsf{ok}$ $\qquad \dfrac{\Gamma(x) = \tau \quad \Gamma \vdash e : \tau}{\Gamma \vdash x \leftarrow e\ \mathsf{ok}}\,(0\text{-}1) \qquad \dfrac{\Gamma \vdash s_1\ \mathsf{ok} \quad \Gamma \vdash s_2\ \mathsf{ok}}{\Gamma \vdash s_1 ; s_2\ \mathsf{ok}}\,(0\text{-}2)$

$$\dfrac{\Gamma \vdash e : \mathsf{bool} \quad \Gamma \vdash s\ \mathsf{ok}}{\Gamma \vdash \mathsf{while}\ e\ s\ \mathsf{ok}}\,(0\text{-}3) \qquad \dfrac{\Gamma \vdash e : \mathsf{bool} \quad \Gamma \vdash s_1\ \mathsf{ok} \quad \Gamma \vdash s_2\ \mathsf{ok}}{\Gamma \vdash \mathsf{if}\ e\ s_1\ \mathsf{else}\ s_2\ \mathsf{ok}}\,(0\text{-}4) \qquad \dfrac{}{\Gamma \vdash \mathsf{skip}\ \mathsf{ok}}\,(0\text{-}5)$$

$x \leftarrow \bar{0};$
$y \leftarrow \bar{5};$
$\mathsf{while}\ (\bar{0} < y)$
$\quad x \leftarrow x + \bar{1};$
$\quad y \leftarrow y - \bar{1}$

$\mapsto$

$y \leftarrow 5;$
$\mathsf{while}\ (\bar{0} < y)$
$\quad x \leftarrow x + \bar{1};$
$\quad y \leftarrow y - \bar{1}$

$[x \mapsto 0]$

Need a "memory"

$\mapsto$

$\cancel{x}\ \bar{0} < \bar{5}$

$\mathsf{while}\ (\bar{0} < y)$
$\quad x \leftarrow x + \bar{1}$
$\quad y \leftarrow y - 1$

$[x \mapsto 0,\ y \mapsto 5]$

$x \leftarrow \bar{0} + \bar{1}$

e val $\qquad e \mapsto_\sigma e \qquad \dfrac{\sigma(x) = v}{x \mapsto_\sigma v}$ Otherwise same

New judgement: $\underbrace{\overset{\swarrow \text{store}}{\sigma}; s}_{\text{"State"}} \mapsto \sigma; s$

$$\frac{e \mapsto_\sigma e'}{\sigma; x \leftarrow e \mapsto \sigma; x \leftarrow e'}\ (\text{ss-1}) \qquad \frac{e \text{ val}}{\sigma; x \leftarrow e \mapsto \sigma[x \mapsto e]; \text{skip}}\ (\text{ss-2})$$

$$\frac{\sigma; s_1 \mapsto \sigma'; s_1'}{\sigma; s_1; s_2 \mapsto \sigma'; s_1'; s_2}\ 3 \qquad \frac{}{\sigma; \text{skip}; s \mapsto \sigma; s}\ 4$$

$$\frac{e \mapsto_\sigma e'}{\sigma; \text{if } e\ s_1 \text{ else } s_2 \mapsto \sigma; \text{if } e'\ s_1 \text{ else } s_2}\ 5 \qquad \frac{}{\sigma; \text{if true } s_1 \text{ else } s_2 \mapsto \sigma; s_1}\ 6$$

$$\frac{}{\sigma; \text{if false } s_1 \text{ else } s_2 \mapsto \sigma; s_2}\ 7$$

$$\frac{}{\sigma; \text{while } e\ s \mapsto \sigma; \text{if } e\ (s; \text{while } e\ s)\ \text{skip}}\ 8$$

$\Gamma = x : \text{int} \qquad\qquad \Gamma = x : \text{int}$

$\underset{\text{int}}{x} \mapsto_{[x \mapsto \text{true}]} \underset{\text{bool}}{\text{true}}$ $\times$ $\qquad x \not\mapsto_{[\,]}$ $\times$

$\boxed{\Gamma \vdash \sigma} = \forall x : \tau \in \Gamma,\ x \in \text{dom}(\sigma)$ and $\Gamma \vdash \sigma(x) : \tau$.

Note: 1. All variables, types declared at beginning in Γ

2. Types of variables don't change

Can relax both, but gets complicated

Progress: 1. If $\Gamma \vdash e : \tau$ and $\Gamma \vdash \sigma$ then e val or $e \mapsto_\sigma e'$
2. If $\Gamma \vdash s$ ok and $\Gamma \vdash \sigma$ then $s = \text{skip}$ or $\sigma; s \mapsto \sigma'; s'$

Proof: 1. $\dfrac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$ By def, $\sigma(x) = v$
Have $x \mapsto_\sigma v$.

2. (O-1) Then $e = x \leftarrow e_0$ and $\Gamma \vdash e_0 : \tau$.
By (1), e_0 val or $e_0 \mapsto e_0'$.
(e_0 val) then apply (SS-2)
($e_0 \mapsto e_0'$) then apply (SS-1)

(O-2) Then $e = s_1; s_2$ and $\Gamma \vdash s_1$ ok and $\Gamma \vdash s_2$ ok.
By induction, $s_1 = \text{skip}$ or $\sigma; s_1 \mapsto \sigma'; s_1'$.
($s_1 = \text{skip}$) Apply (SS-4)
($\sigma; s_1 \mapsto \sigma'; s_1'$) Apply (SS-3)

(O-3) Apply (SS-8)

Preservation:

1. If $\Gamma \vdash e : \tau$ and $\Gamma \vdash \sigma$ and $e \mapsto_\sigma e'$ then $\Gamma \vdash e' : \tau$
2. If $\Gamma \vdash s$ ok and $\Gamma \vdash \sigma$ and $\sigma ; s \mapsto \sigma' ; s'$ then $\Gamma \vdash s'$ ok and $\Gamma \vdash \sigma'$

1. $\dfrac{\sigma(x) = v}{x \mapsto_\sigma v}$ By inversion, $\Gamma(x) = \tau$. By defn $\Gamma \vdash v : \tau$.

2. (SS-1) Then $s = x \leftarrow e$ and $s' = x \leftarrow e'$ and $\sigma = \sigma'$ and $e \mapsto_\sigma e'$.
   By inversion on (O-1), $\Gamma(x) = \tau$ and $\Gamma \vdash e : \tau$.
   By (1), $\Gamma \vdash e' : \tau$.
   Apply (O-1).

   (SS-2) Then $s = x \leftarrow v$ and v val and $s' = \text{skip}$ and $\sigma' = \sigma[x \mapsto v]$
   By inversion on (O-1), $\Gamma(x) = \tau$ and $\Gamma \vdash v : \tau$. So $\Gamma \vdash \sigma'$
   By (O-5), $\Gamma \vdash \text{skip}$ OK.

   (SS-8) Then $s = \text{while } e\ s_0$ and $s' = \text{if } e\ (s_0 ; \text{while } e\ s_0)\ \text{skip}$ and $\sigma' = \sigma$.
   By inversion on (O-3), $\Gamma \vdash e : \text{bool}$ and $\Gamma \vdash s_0$ ok
   By assumption, $\Gamma \vdash \text{while } e\ s_0$ ok.
   By (O-2) $\Gamma \vdash s_0 ; \text{while } e\ s_0$ ok
   By (O-5), $\Gamma \vdash \text{skip}$ ok
   By (O-4), $\Gamma \vdash s'$ ok