Modal Logic

Before: $\Gamma \vdash A$ true
Modal logics $\Gamma \vdash A$ "neccessary" "possible" "sometime" "always"...

This formulation:
Pfenning + Davies, 2000

Lax logic: (Mendler 1990, 93)
$b > 0 \to a + b > a$ true under some constraints
(no overflow,
hardware correct,
no cosmic rays...)

$\Gamma \vdash A$ lax

Can we mix lax and true?

$$\frac{\Gamma \vdash A \text{ true}}{\Gamma \vdash A \text{ lax}}$$

Not reverse, but...

If A true under some constraints, then
"A is true under some constraints" is true.
(under some constraints = $\bigcirc$)

$$\frac{\Gamma \vdash A \text{ lax}}{\Gamma \vdash \bigcirc A \text{ true}} \qquad \frac{\Gamma \vdash \bigcirc A \text{ true} \quad \Gamma, A \text{ true} \vdash B \text{ lax}}{\Gamma \vdash B \text{ lax}}$$

$$\cfrac{\cfrac{\cfrac{\cdot \vdash b > 0 \text{ true}}{\cdot \vdash b > 0 \text{ lax}}}{\cdot \vdash \bigcirc(b > 0) \text{ true}} \quad b > 0 \vdash a + b > a \text{ lax}}{\cdot \vdash a + b > a \text{ lax}}$$

As a type system:

Expressions $e ::= x \mid () \mid \lambda x.e \mid e\,e \mid \ldots$ ← pure

Commands $m ::= \,!e \mid e := e$

(! ← read, := ← write)

$$\frac{\Gamma \vdash m \div \tau}{\Gamma \vdash \text{cmd}(m) : \tau\ \text{cmd}} \qquad \frac{\Gamma \vdash e : \tau_1\ \text{cmd} \quad \Gamma, x : \tau_1 \vdash m \div \tau_2}{\Gamma \vdash x \leftarrow e; m \div \tau_2}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{return } e \div \tau}$$

Lets us pass around commands but still treat exprs as pure

$e ::= \ldots \mid \text{cmd } m$

$m ::= \ldots \mid \text{return } e \mid x \leftarrow e; m$

$$\frac{}{\text{cmd } m \text{ val}} \qquad \frac{e \mapsto e'}{\sigma; \text{return } e \mapsto \sigma; \text{return } e'} \qquad \frac{e \mapsto e'}{\sigma; x \leftarrow e; m \mapsto \sigma; x \leftarrow e'; m}$$

$$\frac{\sigma; m_1 \mapsto \sigma'; m_1'}{\sigma; x \leftarrow \text{cmd } m_1; m_2 \mapsto \sigma; x \leftarrow \text{cmd } m_1'; m_2} \qquad \frac{v \text{ val}}{\sigma; x \leftarrow \text{cmd}(\text{return } v); m_2 \mapsto \sigma; [v/x]m_2}$$

$$\frac{}{\sigma; !\ell \mapsto \sigma; \text{return } \sigma(\ell)} \qquad \frac{}{\sigma; \ell := v \mapsto \sigma[\ell \mapsto v]; \text{return } ()}$$

$$\begin{array}{l} \_ \leftarrow \text{cmd}\ (\ell := 5); \\ x \leftarrow \text{cmd}\ (!\ell); \\ x+2 \end{array} \mapsto [\ell \mapsto 5]\ \begin{array}{l} x \leftarrow \text{cmd}\ (!\ell); \\ x+2 \end{array} \mapsto^2 [\ell \mapsto 5]\ \begin{array}{l} x \leftarrow \text{cmd}\ (\text{return } 5); \\ x+2 \end{array}$$

$$\mapsto [\ell \mapsto 5]\ 5+2 \mapsto 7$$

$$\begin{array}{l} add_n \leftarrow \text{cmd}\ (\text{return } \lambda n.\ \text{cmd}\ (\ell := !\ell + n)) \\ \quad \_ \leftarrow add_n\ 2 \end{array}$$

$$\mapsto \_ \leftarrow (\lambda n.\ \text{cmd}\ (\ell := !\ell + n))\ 2$$

$$\mapsto \_ \leftarrow \text{cmd}\ (\ell := !\ell + 2)$$

$$\mapsto \ldots$$

More modalities

In linear logic: $!A$ = A true w/no linear assumptions

$\Box A$ "A is necessary" $\Diamond A$ "A is possible"
What do these mean?

Possible worlds: $\Gamma \vdash A\ @\ w$ "A is true at w"

$\Gamma \vdash$ it is raining @ Chicago
$\Gamma \vdash$ it is raining @ today

Diff. choices give diff. modal logics

One meaning: $\dfrac{\forall w' \in W.\ \Gamma \vdash A\ @\ w'}{\Gamma \vdash \Box A\ @\ w}$ "it is raining everywhere"

Another: $\dfrac{\forall w' \geq w.\ \Gamma \vdash A\ @\ w'}{\Gamma \vdash \Box A\ @\ w}$ ($\geq$: later in time) "it will always be raining"