Polymorphism

Poly = many, Morph = shape, form

Consider $\lambda x{:}\text{unit}.\, x : \text{unit} \to \text{unit}$

$\lambda x{:}\text{unit} \times \text{unit}.\, x : (\text{unit} \times \text{unit}) \to (\text{unit} \times \text{unit})$

$\lambda x{:}\text{int}.\, x : \text{int} \to \text{int}$

...

type var. polymorphic type

$\tau ::= \dots \mid \alpha \mid \forall \alpha.\, \tau$

$e ::= \dots \mid \Lambda \alpha.\, e \mid e[\tau]$ ← apply to a type

function that takes a type!

Polymorphic identity function: $\underbrace{\Lambda \alpha.\, \lambda (x{:}\alpha).\, x}_{\text{id}} : \forall \alpha.\, \alpha \to \alpha$

Statics: $\Delta; \Gamma \vdash e : \tau$

ctx of type vars in scope

$$\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda \alpha.\, e : \forall \alpha.\, \tau} \qquad \frac{\Delta; \Gamma \vdash e : \forall \alpha.\, \tau'}{\Delta; \Gamma \vdash e[\tau] : [\tau/\alpha]\tau'}$$

$\text{id}[\text{unit}] : \text{unit} \to \text{unit}$

$\text{id}[\text{int}] : \text{int} \to \text{int}$

...

Dynamics: $$\frac{}{\Lambda \alpha.\, e \ \text{val}} \qquad \frac{}{(\Lambda \alpha.\, e)[\tau] \mapsto [\tau/\alpha]e}$$

$$\frac{\cdot \vdash \text{inl}\ () : \text{unit} + \text{int} \times \text{int list}}{\cdot \vdash \text{fold}_{\text{int list}}\ \text{inl}\ () : \mu\alpha.\ \text{unit} + \text{int} \times \text{int list} \triangleq \text{int list}}$$

$\text{il} \triangleq \text{fold}_{\text{int list}}\ \text{inl}\ ()$

$\text{ns}\ e_1^{:\text{int}}\ e_2^{:\text{int list}} \triangleq \text{fold}_{\text{int list}}\ \text{inr}\ (e_1, e_2)$

$\text{d\_or\_0} : \text{int list} \to \text{int}$

$\triangleq \lambda l : \mu\alpha.\ \text{unit} + \text{int} \times \alpha.\ \text{case unfold}\ l\ \text{of}$
$\quad \{x.\ 0;$
$\quad\ y.\ \text{fst}\ y\}$

$$\frac{\Gamma, l : \mu\alpha.\text{unit} + \text{int} \times \alpha \vdash l : \mu\alpha.\text{unit} + \text{int} \times \alpha}{\Gamma, l : \dots \vdash \text{unfold}\ l : [\mu\alpha.\text{unit} + \text{int} \times \alpha / \alpha](\text{unit} + \text{int} \times \alpha)}$$

$= [\text{int list} / \alpha](\text{unit} + \text{int} \times \alpha)$

$= \text{unit} + \text{int} \times \text{int list}$

$$\frac{e\ \text{val}}{\text{ld}_\tau\ e\ \text{val}} \qquad \frac{e \mapsto e'}{\text{fold}_\tau\ e \mapsto \text{fold}_\tau\ e'} \qquad \frac{e \mapsto e'}{\text{unfold}\ e \mapsto \text{unfold}\ e'} \qquad \frac{e\ \text{val}}{\text{unfold}(\text{fold}_\tau\ e) \mapsto e}$$

General Recursion

$\text{sum} : \text{int list} \to \text{int}$

$\triangleq \lambda l : \text{int list}.\ \text{case unfold}\ l\ \text{of}$
$\quad \{\_.\ 0;$
$\quad\ x.\ (\text{fst}\ x) + \text{sum}\ (\text{snd}\ x)\}$

oops

need another fixed pt combinator

let's just add it.

$e ::= \cdots \mid \text{fix}\ x.e$

$$\frac{\Gamma, x : \tau \vdash e : \tau}{\Gamma \vdash \text{fix}\ x.e : \tau} \qquad \frac{}{\text{fix}\ x.e \mapsto [\text{fix}\ x.e / x]e}$$

Can combine this with other STLC features

$\Lambda\alpha.\Lambda\beta.\lambda(x:\alpha\times\beta).(\text{snd } x, \text{fst } x) : \forall\alpha.\forall\beta.\alpha\times\beta\to\beta\times\alpha$

<u>Parametricity</u>

Q: How many <u>distinct</u> values are there of type $\forall\alpha.\alpha\to\alpha$?

Distinct = not observationally equivalent ($\cong$)

Observational equivalence (roughly):

$e_1 \cong e_2$ iff evaluate to same answer in all contexts.

e.g. $\lambda x.e_1 \cong \lambda x.e_2$ iff $\forall e$, $(\lambda x.e_1)e \cong (\lambda x.e_2)e$

A: Just one: id.

Parametricity: In $\Lambda\alpha.e$, e can't "inspect" α.

In $\Lambda\alpha.\lambda(x:\alpha).e$, e can't do anything w/ x except pass it around.

What about $\forall\alpha.\forall\beta.\alpha\times\beta\to\beta\times\alpha$? Also one.

$\forall\alpha.\alpha$? None.

$\forall\alpha.\forall\beta.\alpha\times\beta\to\alpha+\beta$? Two.

$\forall\alpha.\alpha\times\alpha\to\alpha$? Two.

Can also <u>not</u> combine it w/ (most) other STLC features

System F $\quad \tau ::= \alpha \mid \tau \to \tau \mid \forall \alpha . \tau$

$\qquad\qquad e ::= x \mid \lambda x . e \mid e\, e \mid \Lambda \alpha . e \mid e[\tau]$

Surprisingly powerful!

$$\text{unit} \triangleq \forall \alpha . \alpha \to \alpha \qquad () \triangleq \Lambda \alpha . \lambda (x : \alpha) . x$$

$$\text{void} \triangleq \forall \alpha . \alpha \qquad \text{abort } e \triangleq e[\tau]$$

$$\tau_1 \times \tau_2 \triangleq \forall \alpha . (\tau_1 \to \tau_2 \to \alpha) \to \alpha$$

$$(e_1, e_2) \triangleq \Lambda \alpha . \lambda (x : \tau_1 \to \tau_2 \to \alpha) . x\, e_1\, e_2$$

$$\text{fst } e \triangleq e[\tau_1] (\lambda (x : \tau_1) . \lambda (y : \tau_2) . x)$$

$$\text{snd } e \triangleq e[\tau_2] (\lambda (x : \tau_1) . \lambda (y : \tau_2) . y)$$

$$\tau_1 + \tau_2 \triangleq \forall \alpha . (\tau_1 \to \alpha) \to (\tau_2 \to \alpha) \to \alpha$$

$$\text{inl } e \triangleq \Lambda \alpha . \lambda (f : \tau_1 \to \alpha) . \lambda (g : \tau_2 \to \alpha) . f\, e$$

$$\text{inr } e \triangleq \Lambda \alpha . \lambda (f : \tau_1 \to \alpha) . \lambda (g : \tau_2 \to \alpha) . g\, e$$

$$\text{case } e \text{ of } \{x . e_1 ; y . e_2\} \triangleq e[\tau] (\lambda (x : \tau_1) . e_1) (\lambda (y : \tau_2) . e_2)$$

Jean-Yves Girard $\qquad\qquad$ John Reynolds