Simply Typed Lambda Calculus (STLC)

$\tau ::= \text{unit} \mid \tau \to \tau \mid \tau \times \tau \mid \tau + \tau \mid \text{void}$

$e ::= x \mid () \mid e\ e \mid \lambda x{:}\tau.e \mid (e,e) \mid \text{fst}\ e \mid \text{snd}\ e \mid \text{inl}\ e \mid \text{inr}\ e \mid$
$\quad \text{case}\ e\ \text{of}\ \{x.e;\ y.e\} \mid \text{abort}\ e$

$\Gamma \vdash e : \tau$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ (T\text{-}1) \qquad \frac{}{\Gamma \vdash () : \text{unit}}\ (T\text{-}2) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}\ (T\text{-}3)$$

$$\frac{\Gamma, x{:}\tau \vdash e : \tau'}{\Gamma \vdash \lambda x{:}\tau.e : \tau \to \tau'}\ (T\text{-}4)$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}\ (T\text{-}5) \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{fst}\ e : \tau_1}\ (T\text{-}6) \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \text{snd}\ e : \tau_2}\ (T\text{-}7)$$

$\text{dims} \triangleq \underset{x}{\text{int}} \times \underset{y}{\text{int}} \qquad \text{person} \triangleq \underset{\text{name}}{\text{string}} \times \underset{\text{age}}{\text{int}}$

$\text{test} \triangleq \text{negative} \mid \text{positive}$ ← enum (e.g. Java)

$\text{lolla} : \text{proof} \to \text{bool} \qquad \text{proof}: \text{vaccine of } \overset{\text{vax}}{\text{string}} \times \overset{\text{doses}}{\text{int}}$
$\qquad\qquad\qquad\qquad\qquad\qquad \text{or test of test}$

$\text{proof} \triangleq (\text{string} \times \text{int}) + \text{test} \qquad \text{test} \triangleq \text{unit} + \text{unit}$

$\text{neg\_test} \triangleq \text{inr negative} \qquad \text{negative} \triangleq \text{inl}\ ()$

$\text{pfizer\_vax} \triangleq \text{inl}\ (\text{"Pfizer"}, 2) \qquad \text{positive} \triangleq \text{inr}\ ()$

$\text{lolla} \triangleq \lambda p{:}\text{proof}.\ \text{case}\ p\ \text{of}$
$\quad \{\text{vax}.\ \text{snd vax} = 2 \mid\mid \text{fst vax} = \text{"J\&J"};$
$\quad \text{test}.\ \text{test} = \text{negative}\}$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathrm{inl}\ e : \tau_1 + \tau_2} \ (T\text{-}8) \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathrm{inr}\ e : \tau_1 + \tau_2} \ (T\text{-}9)$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \qquad \Gamma, x:\tau_1 \vdash e_1 : \tau \qquad \Gamma \vdash y:\tau_2 \vdash e_2 : \tau}{\Gamma \vdash \mathrm{case}\ e\ \mathrm{of}\ \{x.e_1 ; y.e_2\} : \tau} \ (T\text{-}10)$$

Notice: Binary product $\tau_1 \times \tau_2$
1 way to create (pair)
2 things to do with it (fst, snd)

n-ary product $\tau_1 \times \ldots \times \tau_n$
1 way to create (tuple)
n things to do (projections)

Binary sum $\tau_1 + \tau_2$
2 ways to create (inl, inr)
1 thing to do (case)

n-ary sum $\tau_1 + \ldots + \tau_n$
n ways to create
1 thing to do

0-ary (nullary) product
1 way to create (supply nothing) = Unit!
0 things to do

nullary sum: 0 ways to create, 1 thing to do "void"

$$\frac{\Gamma \vdash e : \mathrm{void}}{\Gamma \vdash \mathrm{abort}\ e : \tau} \ (T\text{-}11)$$

$$\frac{v\ \text{val}}{(\lambda x{:}\tau.e)\ v \mapsto [v/x]e} \qquad \frac{}{(\lambda x.e)\ \text{val}} \qquad \frac{}{()\ \text{val}} \qquad \frac{v_1\ \text{val} \quad v_2\ \text{val}}{(v_1, v_2)\ \text{val}}$$

$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \qquad \frac{v_1\ \text{val} \quad e_2 \mapsto e_2'}{(v_1, e_2) \mapsto (v_1, e_2')} \qquad \frac{e \mapsto e'}{\text{fst}\ e \mapsto \text{fst}\ e'}$$

$$\frac{v_1\ \text{val} \quad v_2\ \text{val}}{\text{fst}\ (v_1, v_2) \mapsto v_1} \qquad \frac{e \mapsto e'}{\text{inl}\ e \mapsto \text{inl}\ e'} \qquad \frac{v\ \text{val}}{\text{inl}\ v\ \text{val}}$$

$$\frac{e \mapsto e'}{\text{case}\ e\ \text{of}\ \{x.e_1;\ y.e_2\} \mapsto \text{case}\ e'\ \text{of}\ \{x.e_1;\ y.e_2\}}$$

$$\frac{}{\text{case inl}\ v\ \text{of}\ \{x.e_1;\ y.e_2\} \mapsto [v/x]e_1}$$

$$\frac{}{\text{case inr}\ v\ \text{of}\ \{x.e_1,\ y.e_2\} \mapsto [v/y]e_2}$$

Progress: If $\Gamma \vdash e : \tau$ then e val or $e \mapsto e'$
Preservation: If $\Gamma \vdash e : \tau$ and $e \mapsto e'$ then $\Gamma \vdash e' : \tau$.

Fun fact: If $\bullet \vdash e : \tau$ then $\exists v$ s.t. v val and $e \mapsto^* v$
(remember: not true for untyped λcalculus)

$$\frac{\dfrac{\dfrac{\Gamma, x{:}\tau \vdash x : \tau \to \tau' \quad \Gamma, x{:}\tau \vdash x : \tau}{\Gamma, x{:}\tau \vdash x\ x : \tau'}}{\Gamma \vdash \lambda x.x\,x : \tau \to \tau'}}{\Gamma \vdash (\lambda x.x\,x)\ (\lambda x.x\,x) :} \qquad \tau = \tau \to \tau' \quad \text{WAT?}$$