Lecture 4

9/1

$e ::= \ldots \mid \text{let } x = e \text{ in } e \mid x$

$\text{let } x = e_1 \text{ in } e_2$ Var. x is **bound** in e_2 (but not e_1)

$\text{let } x = 1 \text{ in } x + 2$

If a variable isn't bound, it's free.
$FV(e)$ = Free variables of e

$FV(x) = \{x\}$
$FV(\bar{n}) = FV('s') = \emptyset$
$FV(e_1 + e_2) = FV(e_1 \mathbin{\hat{}} e_2) = FV(e_1) \cup FV(e_2)$
$FV(|e|) = FV(e)$
$FV(\text{let } x = e_1 \text{ in } e_2) = FV(e_1) \cup (FV(e_2) \setminus \{x\})$

$\text{let } x = y \text{ in } x + 2$

y is free

α-conversion: Can always (consistently) rename bound vars.
α-equivalent: expressions are the same up to α-conversion.

$\text{let } x = 1 \text{ in } x + 2 \equiv_\alpha \text{let } y = 1 \text{ in } y + 2$
$\text{let } x = y \text{ in } x + 2 \not\equiv_\alpha \text{let } x = z \text{ in } x + 2$

let $x = 1$ in $x+2$

$\mapsto$ $1+2$ Substitute 1 for x

$\mapsto$ 3

$[e_1/x]\, e_2$ "substitute e_1 for x in e_2"

$$
\begin{array}{lll}
[e/x]\, x & = e & \\
[e/x]\, y & = y & y \neq x \\
[e/x]\, \bar{n} & = \bar{n} & \\
[e/x]\, \text{"s"} & = \text{"s"} & \\
[e/x](e_1 + e_2) & = [e/x]\, e_1 + [e/x]\, e_2 & \\
[e/x](\text{let } x = e_1 \text{ in } e_2) & = \text{let } x = [e/x]\, e_1 \text{ in } e_2 & \\
[e/x](\text{let } y = e_1 \text{ in } e_2) & = \text{let } y = [e/x]\, e_1 \text{ in } [e/x] e_2 & y \neq x,\ y \notin FV(e)
\end{array}
$$

$[e/x]\,(\text{let } x = 1 \text{ in } x+2) \neq \text{let } x = 1 \text{ in } e+2$

$\equiv_\alpha [e/x]\,(\text{let } y = 1 \text{ in } y+2)$

$[x+2/y]\,(\text{let } x = 1 \text{ in } y+2) \neq \text{let } x = 1 \text{ in } x+2+2$

this x is supposed to be free

now is bound ("captured")

What to do if $y \notin FV(e)$? α-convert

Dynamics

"call-by-value"

$$\frac{e_1 \mapsto e_1'}{\text{let } x = e_1 \text{ in } e_2 \mapsto \text{let } x = e_1' \text{ in } e_2}\ (S\text{-}9) \qquad \frac{v \text{ val}}{\text{let } x = v \text{ in } e_2 \mapsto [v/x]e_2}\ (S\text{-}10)$$

"Call-by-name"

$$\frac{}{\text{let } x = e_1 \text{ in } e_2 \mapsto [e_1/x]e_2}\ (S\text{-}9)$$

Statics

$\Gamma \vdash e : \tau$ "under context Γ, e has type τ"

Γ context: maps vars. to types

$$\frac{}{\Gamma \vdash \bar{n} : \text{int}}\ (T\text{-}1) \qquad \frac{}{\Gamma \vdash \text{"s"} : \text{string}}\ (T\text{-}2) \qquad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}}\ (T\text{-}3)$$

$$\frac{\Gamma \vdash e_1 : \text{string} \quad \Gamma \vdash e_2 : \text{string}}{\Gamma \vdash e_1 \hat{}\ e_2 : \text{string}}\ (T\text{-}4) \qquad \frac{\Gamma \vdash e : \text{string}}{\Gamma \vdash |e| : \text{int}}\ (T\text{-}5)$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ (T\text{-}6)$$

sometimes: $\dfrac{}{\Gamma, x:\tau \vdash x : \tau}$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x:\tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}\ (T\text{-}7)$$

Contexts usually written $x_1:\tau_1, x_2:\tau_2, \ldots, x_n:\tau_n$

$\Gamma, x:\tau$ – extend Γ w/ $x:\tau$ (impl: $x \notin dom(\Gamma)$)

$\emptyset, \bullet$: empty ctx

Structural properties

Weakening: If $\Gamma \vdash e : \tau$ and $x \notin dom(\Gamma)$, then $\Gamma, x:\tau' \vdash e : \tau$

Pf: By induction on the derivation of $\Gamma \vdash e : \tau$ $\square$

Substitution: If $\Gamma, x:\tau' \vdash e : \tau$ and $\Gamma \vdash e' : \tau'$

then $\Gamma \vdash [e'/x]e : \tau$.

Pf: By induction on the derivation of $\Gamma, x:\tau' \vdash e : \tau$.

T-3 $[e'/x]e = [e'/x]e_1 + [e'/x]e_2$

By IH, $\Gamma \vdash [e'/x]e_1 : int$, $\Gamma \vdash [e'/x]e_2 : int$

Apply T-3

T-6

$$\frac{\Gamma(y) = \tau}{\Gamma \vdash y : \tau}$$

Case 1: $x = y$. Then $[e'/x]e = e'$, and $\tau = \tau'$.

By assumption, $\Gamma \vdash e' : \tau'$.

Case 2: $x \neq y$. Then $[e'/x]y = y$ and by T-6, $\Gamma \vdash y : \tau$

T-7 Case 1: $x = y$ Then $[e'/x]e =$ let $x = [e'/x]e_1$ in e_2

By induction, $\Gamma \vdash [e'/x]e_1 : \tau_1$.

We have $\Gamma, x:\tau_1 \vdash e_2 : \tau_2$. Apply rule T-7.

Case 2: $x \neq y$. Then $[e'/x]e =$ let $y = [e'/x]e_1$ in $[e'/x]e_2$

We have $\Gamma, x:\tau' \vdash e_1 : \tau_1$ and $\Gamma, x:\tau', y:\tau_1 \vdash e_2 : \tau_2$

By induction, $\Gamma \vdash [e'/x]e_1 : \tau_1$.

By weakening, $\Gamma, y:\tau_1 \vdash e' : \tau'$.

By induction, $\Gamma, y:\tau_1 \vdash [e'/x]e_2 : \tau_2$.

By T-7, $\Gamma \vdash [e'/x]e : \tau_2$.

Preservation: If $\emptyset \vdash e : \tau$ and $e \mapsto e'$ then $\emptyset \vdash e' : \tau$

S-10 By inversion, $\Gamma \vdash v_1 : \tau_1$ and $\Gamma, x : \tau_1 \vdash e_2 : \tau_2$.

(S-9 for CBN) By substitution, $\Gamma \vdash [v/x] e_2 : \tau_2$. $\square$

Progress: If $\emptyset \vdash e : \tau$ then e val or $e \mapsto e'$

T-> By IH, e_1 val or $e_1 \mapsto e_1'$

then let $x = e_1$ in $e_2 \mapsto [e_1/x] e_2$ by S-10

then let $x = e_1$ in $e_2 \mapsto$ let $x = e_1'$ in e_2 by S-9

(CBN: let $x = e_1$ in $e_2 \mapsto [e_1/x] e_2$)