Satisfaction, Validity, and State Updates

CS 536: Science of Programming, Fall 2021

1. Say u and v stand for variables (possibly the same variable) and α and β are values (possibly equal). When is $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$? Hint: There are four cases because maybe $u \equiv v$ and maybe $\alpha = \beta$.

2. Let $\sigma(b) = (7, 5, 12, 16)$. Assume out-of-bound indexes cause runtime errors.
   a. Does $\sigma \vDash \exists k \,.\, 0 \le k \wedge k{+}1 < size(b) \wedge b[k] < b[k{+}1]$? If so, what was your witness value for k?
   b. Does $\sigma \vDash \exists k \,.\, 0 < k{-}1 \wedge k{+}1 < size(b) \wedge b[k{-}1] < b[k] < b[k{+}1]$? If so, what was your witness value for k?
   c. Does $\sigma \vDash \forall k \,.\, 0 \le k < 4 \rightarrow b[k] > 0$?
   d. If $\sigma(k) = -5$, then does $\sigma \vDash \exists k \,.\, b[k] > 0$?

3. For each of the situations below, fill in the blanks to describe when the situation holds.
   Fill in _____$_1$ with "*some*", "*every*", or "*this*"
   Fill in _____$_2$ with "*some*" or "*every*"
   Fill in _____$_3$ with "σ(*x*) must *be undefined*", "σ(*x*) must *be defined and* $\sigma \vDash p$", or "*nothing of* σ(*x*)"
   Fill in _____$_4$ with "$\vDash p$" or "$\nvDash p$"
   a. $\sigma \vDash (\exists x \in U.\ p)$ iff for _____$_1$ state σ and _____$_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____$_4$
   b. $\sigma \vDash (\forall x \in U.\ p)$ iff for _____$_1$ state σ and _____$_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____$_4$
   c. $\sigma \vDash (\exists x \in U.\ p)$ requires _____$_3$.
   d. $\sigma \vDash (\forall x \in U.\ p)$ requires _____$_3$.
   e. $\sigma \nvDash (\exists x \in U.\ p)$ iff for _____$_1$ state σ for _____$_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____$_4$
   f. $\sigma \nvDash (\forall x \in U.\ p)$ iff for _____$_1$ state σ for _____$_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____$_4$
   g. $\nvDash (\forall x \in U.\ p)$ iff for _____$_2$ state σ, we have σ _____$_4$ $(\forall x \in U.\ p)$.
   h. $\nvDash (\exists x \in U.\ p)$ iff for _____$_2$ state σ, we have σ _____$_4$ $(\exists x \in U.\ p)$.
   i. $\nvDash (\forall x \in U.\ p)$ iff for _____$_2$ state σ, and for _____$_2$ $\alpha \in U$, we have $\sigma[x \mapsto \alpha]$ _____$_4$
   j. $\vDash (\exists x \in U\ .\ (\forall y \in V\ .\ p))$ iff for _____$_1$ state σ, for _____$_2$ $\alpha \in U$, and for _____$_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____$_4$
   k. $\nvDash (\exists x \in U\ .\ (\forall y \in V\ .\ p))$ iff for _____$_1$ state σ, for _____$_2$ $\alpha \in U$, and for _____$_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ $[\vDash \mid \vDash \neg]$ p.

l. $\vDash (\forall\, x \in U\ .\ (\exists\, y \in V\ .\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ $_4$

m. $\nvDash (\forall\, x \in U\ .\ (\exists\, y \in V\ .\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ $_4$

4. Let $p \equiv \exists\, y\ .\ \forall\, x\ .\ f(x) > y$, and let $q \equiv \forall\, x\ .\ \exists\, y\ .\ f(x) > y$. (As usual, assume a domain of $\mathbb{Z}$.)

   a. Is it the case that (regardless of the definition of f), if p is valid then so is q? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash p$ but $\nvDash q$.

   b. (The converse.) Is it the case that (regardless of the definition of f), if q is valid then so is p? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash q$ but $\nvDash p$.

CS 536: Solution to Activity 4 (Satisfaction, Validity, and State Updates)

1. $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$ iff $u \not\equiv v$ or ($u \equiv v$ and) $\alpha = \beta$. Another way to phrase this is ($\alpha = \beta$ or $u \not\equiv v$)

2. (Quantified statements over arrays) Let $\sigma(b) = (7, 5, 12, 16)$.
   a. Yes, $\sigma \vDash \exists k \,.\, 0 \le k \wedge k+1 < size(b) \wedge b[k] < b[k+1]$ with 1 and 2 as possible witnesses for k.
   b. Yes, $\sigma \vDash \exists k \,.\, 0 \le k\text{-}1 \wedge k+1 < size(b) \wedge b[k\text{-}1] < b[k] < b[k+1]$ with 2 as the only witness that works.
   c. Yes, $\sigma \vDash \forall k \,.\, b[k] > 0$
   d. Yes, if $\sigma(k) = -5$, we still have $\sigma \vDash \exists k \,.\, b[k] > 0$, with witnesses 0, 1, 2, 3. The key is that for σ to satisfy the existential with witness call it α, then we need $\sigma[k \mapsto \alpha] \vDash b[k] > 0$, which doesn't depend on $\sigma(k)$ because the update of σ uses $k = \alpha$, not k = whatever $\sigma(k)$ happens to be. Here's a step-by-step explanation (this is way too much detail for appearing on a test):

$\sigma[k \mapsto \alpha] \vDash b[k] >$	
iff $\sigma[k \mapsto \alpha](b[k]) > \sigma[k \mapsto \alpha](0)$	defn state ⊨ relational test
iff $(\sigma[k \mapsto \alpha](b))(\sigma[k \mapsto \alpha](k)) > 0$	the value of 0 is zero
iff $(\sigma(b))(\sigma[k \mapsto \alpha](k)) > 0$	$\sigma[k \mapsto \alpha](b) = \sigma(b)$ because $b \not\equiv k$
iff $(\sigma(b))(\alpha) > 0$	$\sigma[k \mapsto \alpha](k)) = \alpha$
iff 7, 5, 12, or 16 > 0	depending on α = 0, 1, 2, or 3

3. (Validity/invalidity of quantified predicates)
   a. this σ, some α, $\vDash p$
   b. this σ, every α, $\vDash p$
   c. nothing of $\sigma(x)$
   d. nothing of $\sigma(x)$
   e. this σ, every α, $\nvDash p$
   f. this σ, some α, $\nvDash p$
   g. some σ, $\nvDash \forall x \in U.\ p$
   h. some σ, every α, $\nvDash p$
   i. some σ, some α, $\nvDash p$
   j. every σ, some α, every β, $\vDash p$
   k. some σ, every α, some β, $\nvDash p$
   l. every σ, every α, some β, $\vDash p$
   m. some σ, some α, every β, $\nvDash p$

4. ($\exists\forall$ predicates versus $\forall\exists$ predicates, specifically $p \equiv \exists y\,.\,\forall x\,.\,f(x) > y$, and $q \equiv \forall x\,.\,\exists y\,.\,f(x) > y$)

   a. The relation does hold: $\vDash p$ implies $\vDash q$. The short explanation is that for each value α for x, we need to find a value β for y that satisfies the body, but p says that there's a value that works for every α, so we can use that value for β. In more detail, assume p is valid: for every state σ, there is some value β where for every value α, $\sigma[y \mapsto \beta][x \mapsto \alpha] \vDash f(x) > y$. To show that q is valid, take an arbitrary state τ with value α for x. We need a witness value for the $\exists y$; using p with τ for σ, we get a β for the $\exists y$ of p and use that as the witness for the $\exists y$ in q. So then we need $\tau[x \mapsto \alpha][y \mapsto \beta] \vDash f(x) > y$. Substituting σ for τ and swapping the order of the updates, we need $\sigma[y \mapsto \beta][x \mapsto \alpha] \vDash f(x) > y$. But that's exactly what p provided.

   b. The relation does not hold: We can have $\vDash q$ but $\nvDash p$. The easiest example is $f(x) = x$, then validity of p would require us to find an integer value for y that is > every possible integer value of x, but no such value exists.

      As an aside, you can use an arbitrary predicate over x and y instead of $f(x) > y$ as the body of the $\exists\forall$ and $\forall\exists$ predicates. I use $f(x) > y$ here just because it's nice and concrete.

