Weakest Preconditions

Part 2: Calculating wp, wlp; Domain Predicates

CS 536: Science of Programming, Spring 2022

For Problems 2 - 4, just syntactically calculate the \( wp \); don't also logically simplify the result.

2. Calculate the \( wp \) in each of the following cases.
   a. \( wp(k := k - s, n = 3 \land k = 4 \land s = -7) \).
   b. \( wp(n := n*(n-k), n = 3 \land k = 4 \land s = -7) \).
   c. \( wp(n := n*(n-k) \land k := k - s, n > k + s) \).

3. Let \( Q(k, s) \equiv 0 \leq k \leq n \land s = \text{sum}(0, k) \) where \( \text{sum}(u, v) \) is the sum of \( u, u+1, ..., v \) (when \( u \leq v \)) or 0 (when \( u > v \)).
   a. Calculate \( wp(k := k+1; s := s+k, Q(k, s)) \).
   b. Calculate \( wp(s := s+k+1; k := k+1, Q(k, s)) \).
   c. Calculate \( wp(s := s+k; k := k+1, Q(k, s)) \). (This one isn't compatible with \( s = \text{sum}(0, k) \).

4. Calculate the \( wp \) below.
   a. \( wp(if\ e\ then\ \{x := x/2\} else\ \{\text{skip}\};\ y := x, x = 5 \land y = z) \).
   b. \( wp(if\ x \geq 0\ then\ \{x := x*2\} else\ \{x := y\};\ x := c*x, a \leq x < y) \)

For Problems 5 and 6, don’t forget the domain predicates. You can logically simplify as you go.

5. Calculate \( p \) to be the \( wp \) in \( \{p\} x := y/b[k] \{x > 0\} \).

6. Calculate \( p_1 \) and \( p_2 \) to be the \( wp \)'s in \( \{p_1\} y := \sqrt{b[k]} \{z < y\} \) and \( \{p_2\} k := x/k \{p_1\} \).
Solution to Practice 11 (Weakest Preconditions, pt. 2)

2. (Calculate \( wlp \))
   a. \( wlp(k := k - s, n = 3 \land k = 4 \land s = -7) \equiv n = 3 \land k - s = 4 \land s = -7 \)
   b. \( wlp(n := n^*(n-k), n = 3 \land k = 4 \land s = -7) \equiv n^*(n-k) = 3 \land k = 4 \land s = -7 \)
   c. \( wlp(n := n^*(n-k); k := k-s, n > k+s) \)
      \[ \equiv wlp(n := n^*(n-k), wlp(k := k-s, n > k+s)) \]
      \[ \equiv wlp(n := n^*(n-k), n > k-s+s) \]
      \[ \equiv n^*(n-k) > k-s+s \]

3. (wp involving sums) We have \( Q(k, s) \equiv 0 \leq k \leq n \land s = \text{sum}(0, k) \).
   a. \( wp(k := k+1; s := s+k, Q(k, s)) \)
      \[ \equiv wp(k := k+1, wp(s := s+k, Q(k, s))) \]
      \[ \equiv wp(k := k+1, Q(k, s+k)) \]
      \[ \equiv wp(k := k+1, 0 \leq k \leq n \land s+k = \text{sum}(0, k)) \]
      \[ \equiv 0 \leq k+1 \leq n \land s+k+1 = \text{sum}(0, k+1) \]
   b. \( wp(s := s+k+1; k := k+1, Q(k, s)) \)
      \[ \equiv wp(s := s+k+1, wp(k := k+1, Q(k, s))) \]
      \[ \equiv wp(s := s+k+1, Q(k+1, s)) \]
      \[ \equiv wp(s := s+k+1, 0 \leq k+1 \leq n \land s = \text{sum}(0, k+1)) \]
      \[ \equiv 0 \leq k+1 \leq n \land s+k+1 = \text{sum}(0, k+1) \]
   c. \( wp(s := s+k; k := k+1, Q(k, s)) \)
      \[ \equiv wp(s := s+k, wp(k := k+1, Q(k, s))) \]
      \[ \equiv wp(s := s+k, Q(k+1, s)) \]
      \[ \equiv wp(s := s+k, 0 \leq k+1 \leq n \land s = \text{sum}(0, k+1)) \]
      \[ \equiv 0 \leq k+1 \leq n \land s+k = \text{sum}(0, k+1). \text{ Note this isn't compatible with } s = \text{sum}(0, k). \]

4. (wp of if-then)
   a. \( wp(\text{if } e \text{ then } \{ x := x/2 \} \text{ else } \{ \text{skip} \}; y := x, x = 5 \land y = z) \)
      \[ \equiv wp(\text{if } e \text{ then } \{ x := x/2 \} \text{ else } \{ \text{skip} \}, wp(y := x, x = 5 \land y = z)) \]
      \[ \equiv wp(\text{if } e \text{ then } \{ x := x/2 \} \text{ else } \{ \text{skip} \}, x = 5 \land x = z) \]
      \[ \equiv (e \rightarrow wp(x := x/2, x = 5 \land x = z)) \land (\neg e \rightarrow wp(\text{skip}, x = 5 \land x = z)) \]
      \[ \equiv (e \rightarrow x/2 = 5 \land x/2 = z) \land (\neg e \rightarrow x = 5 \land x = z) \]
b. $wp(if \ x \geq 0 \ then \ \{x := x*2\} \ else \ \{x := y\}; \ x := c*x, \ a \leq x < y)$. 
   $\equiv wp(S, wp(x := c*x, \ a \leq x < y))$ where $S$ is the $if$ statement
   $\equiv wp(S, \ a \leq c*x < y)$
   $\equiv wp(if \ x \geq 0 \ then \ \{x := x*2\} \ else \ \{x := y\}, \ a \leq c*x < y)$
   $\equiv (x \geq 0 \Rightarrow wp(x := y, \ a \leq c*x < y)) \land (x < 0 \Rightarrow wp(x := y, \ a \leq c*x < y))$
   $\equiv (x \geq 0 \Rightarrow a \leq c*(x*2) < y) \land (x < 0 \Rightarrow a \leq c*y < y)$

5. For $\{p\} \ x := y/b[k] \ \{x > 0\}$,
   let $p \iff wp(x := y/b[k], \ x > 0)$
   $\equiv wp(x := y/b[k], \ x > 0) \land D(x := y/b[k])$
   $\equiv y/b[k] > 0 \land b[k] \neq 0 \land D(b[k])$
   $\equiv y/b[k] > 0 \land b[k] \neq 0 \land 0 \leq k < size(b)$

6. For $\{p_1\} \ y := sqrt(b[k]) \ \{z < y\}$
   let $p_1 \iff wp(y := sqrt(b[k]), \ z < y)$
   $\equiv wp(y := sqrt(b[k]), \ z < y) \land D(y := sqrt(b[k]))$
   $\equiv z < sqrt(b[k]) \land b[k] \geq 0 \land D(b[k])$
   $\equiv z < sqrt(b[k]) \land b[k] \geq 0 \land 0 \leq D(b[k] < size(b))$

For $\{p_2\} \ k := x/k; \ \{p_1\}$, let
   $p_2 \iff wp(k := x/k, \ p_1)$
   $\equiv wp(k := x/k, \ p_1) \land D(k := x/k)$
   $\equiv p_1[x/k \mid k] \land k \neq 0$
   $\equiv (z < sqrt(b[k]) \land b[k] \geq 0 \land 0 \leq D(b[k] < size(b)) [x/k \mid k] \land k \neq 0$
   $\equiv z < sqrt(b[x/k]) \land b[x/k] \geq 0 \land 0 \leq D(b[x/k] < size(b) \land k \neq 0$