**Big-step Semantics; Runtime Errors**

*CS 536: Science of Programming, Fall 2021*

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**Big-step Semantics**

Problems 1 – 4 are the big-step versions of the similar questions from Practice 5

1. What is
   a. \( M(x := x + 1, \{x = 5\}) \)?
   b. \( M(x := x + 1, \sigma) \)? (Your answer will be symbolic.)
   c. \( (x := x + 1; y := 2 * x, \{x = 5\}) \)?

2. Let \( S \equiv \) if \( x > 0 \) then \( \{x := x + 1\} \) else \( \{y := 2 * x\} \).
   a. Let \( \sigma(x) = 8 \). What is \( M(S, \sigma) \)?
   b. Repeat, if \( \sigma(x) = 0 \).
   c. Repeat, if we don’t know what \( \sigma(x) \) is. (Your answer will be symbolic.)

3. Let \( S \equiv \) if \( x > 0 \) then \( \{x := x / z\} \).
   a. What is \( M(S, \sigma) \) if \( \sigma = \{x = 8, z = 3\} \)? (Don’t forget, integer division truncates)
   b. What is \( M(S, \{x = -2, z = 3\}) \)?

4. Let \( W \equiv \) while \( x < 3 \) \{ \( S \) \} where \( S \equiv \{x := x + 1; y := y * x\} \).
   a. Evaluate the body \( S \) in an arbitrary state \( \tau \) and give \( M(S, \tau) \).
   b. What is \( M(W, \sigma) \) if \( \sigma \models x = 4 \land y = 1 \)?
   c. What is \( M(W, \sigma) \) if where \( \sigma \models x = 1 \land y = 1 \)?

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**Runtime Errors**

5. Let \( S \equiv x := y / b[x] \) and let \( \sigma = \{b = \{3, 0, -2, 4\}, x = \alpha, y = 13\} \). Find all \( \alpha \) such that \( M(S, \sigma) = \{\bot_e\} \). (Remember, integer division truncates.)

6. Repeat the previous problem on \( S \equiv y := y / sqrt(b[x]) \) and \( \sigma = \{b = \{-1, 9, 12, 0\}, x = \alpha, y = 8\} \). Treat \( sqrt \) as returning the truncated integer square root of its argument. (i.e., \( sqrt(0) = 0, sqrt \) of 1 through 3 are all 1, \( sqrt \) of 4 through 8 = 2, etc.)
Solution to Practice 6 (Denotational Semantics; Runtime Errors)

Denotational Semantics

1. (Calculate meanings of programs)
   a. \( M(x := x+1, \{x = 5\}) = \{x = 5\}[x \mapsto \{x = 5\}(x+1)] = \{x = 6\} \)
   b. \( M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto \sigma(x)+1]\} \)
   c. \( M(x := x+1; y := 2*x, \{x = 5\}) = M(y := 2*x, M(x := x+1, \{x = 5\}) \)
      = \{\{x = 6\}[y \mapsto \beta]\} \text{ where } \beta = \{x = 6\}(2*x) = 12
      = \{\{x = 6, y = 12\}\}

2. Let \( S \equiv \) if \( x > 0 \) then \( x := x+1 \) else \( y := 2*x \) fi.
   a. If \( \sigma(x) = 8 \), then \( \sigma(x > 0) = T \), so \( M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\} = \{\sigma[x \mapsto 9]\} \)
   b. If \( \sigma(x) = 0 \), then \( \sigma(x > 0) = F \), so \( M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto \sigma(2*x)]\} = \{\sigma[y \mapsto 0]\} \)
   c. If \( \sigma(x) > 0 \) then \( M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\} \)
      If \( \sigma(x) \leq 0 \) then \( M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto 2 \times \sigma(x)]\}\)

3. Let \( S \equiv \) if \( x > 0 \) then \( x := x/z \) fi \( \equiv \) if \( x > 0 \) then \( x := x/z \) else skip fi
   a. If \( \sigma = \{x = 8, z = 3\} \), then \( \sigma(x > 0) = T \), so \( M(S, \sigma) = M(x := x/z, \sigma) = \{\sigma[x \mapsto \alpha]\} \)
      where \( \alpha = \sigma(x/z) = \sigma[x \mapsto 8/3] = \sigma[x \mapsto 2] \), since integer division truncates.
   b. If \( \sigma = \{x = -2, z = 3\} \) then \( \sigma(x > 0) = F \), so \( M(S, \sigma) = M(skip, \sigma) = \{\sigma\} \).

4. Let \( W \equiv \) while \( x < 3 \) do \( S \) od where \( S \equiv x := x+1; y := y*x \).
   a. For arbitrary \( \tau \),
      \( M(S, \tau) = M(x := x+1; y := y*x, \tau) \)
      = \( M(y := y*x, \tau[x \mapsto \tau(x)+1]) \)
      = \( \{\tau[x \mapsto \tau(x)+1][y \mapsto \alpha]\} \text{ where } \alpha = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1) \)
   b. If \( \sigma \models x = 4 \land y = 1 \), then \( \sigma(x < 3) = F \) so \( M(W, \sigma) = \{\sigma\} \).
   c. If \( \sigma \models x = 1 \land y = 1 \), then \( \sigma(x < 3) = T \) so we have at least one iteration to do. Let \( \sigma_0 = \sigma \), let \( \sigma_1 = M(S, \sigma_0) = \sigma_0(y) \times (\sigma_0(x)+1) \), and let \( \sigma_2 = M(S, \sigma_1) = \sigma_1(y) \times (\sigma_1(x)+1) \). Then
      \( \sigma_0 = \sigma[x \mapsto 1][y \mapsto 1] \)
      \( \sigma_1 = M(S, \sigma_0) = \sigma_0(x \mapsto \sigma_0(x)+1)[y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2] \)
      \( \sigma_2 = M(S, \sigma_1) = \sigma_1(x \mapsto 2+1)[y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6] \)
Since $\sigma_0$ and $\sigma_1 \models x < 3$ but $\sigma_2 \models x \geq 3$, we have $M(W, \sigma) = \{\sigma_2\} = \{\sigma[x \mapsto 3][y \mapsto 6]\}$.

Runtime Errors

5. $M(S, \sigma) = M(x := y/b[x], \sigma) = \{\sigma[x \mapsto y]\}$ where $\gamma = \sigma(y/b[x]) = 13/\sigma(b)(\alpha) = \bot$
   if $\sigma(b)(\alpha) = \bot$ or $\sigma(b)(\alpha) = 0$
   if $(\alpha$ is out of range for $\sigma(b))$ or $\sigma(b)(\alpha) = 0$ \hspace{1cm} ($b[x]$ fails if $x$ is out of range)
   if $(\alpha < 0$ or $\alpha \geq 4)$ or $(\alpha = 1)$ \hspace{1cm} ($b[1]$ is the only element = 0)
   if $\neg(\alpha = 0, 2, \text{or } 3)$

6. $M(S, \sigma) = M(y := y/sqrt(b[x]), \sigma) = \{\sigma[y \mapsto \beta]\}$ where $\beta = (\sigma(y)/sqrt(\gamma)) = (8/sqrt(\gamma))$ and $\gamma = \sigma(b)(\alpha) = \sigma(b)(\alpha)$.
   So $\beta = \bot$ and thus $M(S, \sigma) = \{\sigma[y \mapsto \bot]\} = \{\bot\}$
   if $\gamma = \bot$ or $\gamma < 0$ or $sqrt(\gamma) = 0$ \hspace{1cm} ($b[x]$ fails, $b[x] < 0$, or $sqrt(b[x]) = 0$)
   if $(\alpha$ out of range for $\sigma(b))$ or $\gamma < 0$ or $sqrt(\gamma) = 0$ ($\gamma = \bot$ iff $b[x]$ has a bad index)
   if $(\alpha < 0$ or $\alpha \geq 4)$ or $\gamma = \sigma(b)(\alpha) < 0$ or $sqrt(\gamma) = 0$ \hspace{1cm} ($\sigma(b)$ is of size 4)
   if $(\alpha < 0$ or $\alpha \geq 4)$ or $(\alpha = 0)$ or $sqrt(\gamma) = 0$ \hspace{1cm} (only $b[0] < 0$)
   if $(\alpha < 0$ or $\alpha \geq 4)$ or $(\alpha = 0)$ or $(\alpha = 3)$ \hspace{1cm} (only $sqrt(b[3]) = sqrt(0) = 0$)
   if $(\alpha \leq 0$ or $\alpha \geq 3)$ \hspace{1cm} (combining terms)