# Big-step Semantics; Runtime Errors

## CS 536: Science of Programming, Fall 2021

### Big-step Semantics

Problems 1 – 4 are the big-step versions of the similar questions from Practice 5

1. What is
   a. $M(x := x+1, \{x = 5\})$ ?
   b. $M(x := x+1, \sigma)$ ? (Your answer will be symbolic.)
   c. $\langle\, x := x+1;\ y := 2*x, \{x = 5\}\,\rangle$ ?

2. Let $S \equiv$ if $x > 0$ then $\{x := x+1\}$ else $\{y := 2*x\}$.
   a. Let $\sigma(x) = 8$. What is $M(S, \sigma)$?
   b. Repeat, if $\sigma(x) = 0$.
   c. Repeat, if we don't know what $\sigma(x)$ is. (Your answer will be symbolic.)

3. Let $S \equiv$ if $x > 0$ then $\{x := x/z\}$.
   a. What is $M(S, \sigma)$ if $\sigma = \{x = 8, z = 3\}$? (Don't forget, integer division truncates)
   b. What is $M(S, \{x = -2, z = 3\})$?

4. Let $W \equiv$ while $x < 3$ { $S$ } where $S \equiv x := x+1;\ y := y*x$.
   a. Evaluate the body $S$ in an arbitrary state $\tau$ and give $M(S, \tau)$.
   b. What is $M(W, \sigma)$ if $\sigma \models x = 4 \wedge y = 1$?
   c. What is $M(W, \sigma)$ if where $\sigma \models x = 1 \wedge y = 1$?

### Runtime Errors

5. Let $S \equiv x := y/b[x]$ and let $\sigma = \{b = (3, 0, -2, 4), x = \alpha, y = 13\}$. Find all $\alpha$ such that $M(S, \sigma) = \{\perp_e\}$. (Remember, integer division truncates.)

6. Repeat the previous problem on $S \equiv y := y\ /\ sqrt(b[x])$ and $\sigma = \{b = (-1, 9, 12, 0), x = \alpha, y = 8\}$. Treat *sqrt* as returning the truncated integer square root of its argument. (I.e., $sqrt(0) = 0$, *sqrt* of *1* through *3* are all *1*, *sqrt* of *4* through $8 = 2$, etc.)

 

## *Solution to Practice 6 (Denotational Semantics; Runtime Errors)*

### *Denotational Semantics*

1. (Calculate meanings of programs)
   a. $M(x := x+1, \{x = 5\}) = \{\{x = 5\}[x \mapsto \{x = 5\}(x+1)]\} = \{\{x = 6\}\}$
   b. $M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto \sigma(x)+1]\}$
   c. $M(x := x+1;\ y := 2*x, \{x = 5\})$
   $= M(y := 2*x, M(x := x+1, \{x = 5\})$
   $= M(y := 2*x, \{x = 6\})$ [from part (a)]
   $= \{\{x = 6\}[y \mapsto \beta]\}$ where $\beta = \{x = 6\}(2*x) = 12$
   $= \{\{x = 6, y = 12\}\}$

2. Let $S \equiv$ if $x > 0$ then $x := x+1$ else $y := 2*x$ *fi*.
   *a.* If $\sigma(x) = 8$, then $\sigma(x > 0) = T$, so $M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto 9]\}$
   *b.* If $\sigma(x) = 0$, then $\sigma(x > 0) = F$, so $M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto \sigma(2*x)]\} = \{\sigma[y \mapsto 0]\}$
   *c.* If $\sigma(x) > 0$ then $M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\}$
   If $\sigma(x) \le 0$ then $M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto 2 \times \sigma(x)]\}$

3. Let $S \equiv$ *if* $x > 0$ then $x := x/z$ fi $\equiv$ if $x > 0$ then $x := x/z$ else skip fi
   a. If $\sigma = \{x = 8, z = 3\}$, then $\sigma(x > 0) = T$, so $M(S, \sigma) = M(x := x/z, \sigma) = \{\sigma[x \mapsto \alpha]\}$ where $\alpha = \sigma(x/z) = \sigma[x \mapsto 8/3] = \sigma[x \mapsto 2]$, since integer division truncates.
   b. If $\sigma = \{x = -2, z = 3\}$ then $\sigma(x > 0) = F$, so $M(S, \sigma) = M(skip, \sigma) = \{\sigma\}$.

4. Let $W \equiv$ while $x < 3$ do $S$ od where $S \equiv x := x+1;\ y := y*x$.
   a. For arbitrary $\tau$,
   $M(S, \tau) = M(x := x+1;\ y := y*x, \tau)$
   $= M(y := y*x, \tau[x \mapsto \tau(x)+1])$
   $= \{\tau[x \mapsto \tau(x)+1][y \mapsto \alpha]\}$ where $\alpha = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1)$
   b. If $\sigma \vDash x = 4 \wedge y = 1$, then $\sigma(x < 3) = F$ so $M(W, \sigma) = \{\sigma\}$.
   c. If $\sigma \vDash x = 1 \wedge y = 1$, then $\sigma(x < 3) = T$ so we have at least one iteration to do. Let $\sigma_0 = \sigma$, let $\sigma_1 = M(S, \sigma_0) = \sigma_0(y) \times (\sigma_0(x)+1)$, and let $\sigma_2 = M(S, \sigma_1) = \sigma_1(y) \times (\sigma_1(x)+1)$. Then
   $\sigma_0 = \sigma[x \mapsto 1][y \mapsto 1]$
   $\sigma_1 = M(S, \sigma_0) = \sigma_0[x \mapsto \sigma_0(x)+1][y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2]$
   $\sigma_2 = M(S, \sigma_1) = \sigma_1[x \mapsto 2+1][y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6]$

Since $\sigma_0$ and $\sigma_1 \vDash x < 3$ but $\sigma_2 \vDash x \geq 3$, we have $M(W, \sigma) = \{\sigma_2\} = \{\sigma[x \mapsto 3][y \mapsto 6]\}$.

*Runtime Errors*

5. $M(S, \sigma) = M(x := y/b[x], \sigma) = \{\sigma[x \mapsto \gamma]\}$ *where* $\gamma = \sigma(y/b[x]) = 13/\sigma(b)(\alpha) = \perp$
   iff $\sigma(b)(\alpha) = \perp$ or $\sigma(b)(\alpha) = 0$
   iff ($\alpha$ is out of range for $\sigma(b)$) or $(\sigma(b)(\alpha) = 0)$ ($b[x]$ fails if $x$ is out of range)
   iff $(\alpha < 0$ *or* $\alpha \geq 4)$ or $(\sigma(b)(\alpha) = 0)$ ($\sigma(b)$ has size $4$)
   iff $(\alpha < 0$ *or* $\alpha \geq 4)$ or $(\alpha = 1)$ ($b[1]$ is the only element $= 0$)
   iff $\neg(\alpha = 0, 2,$ or $3)$

6. $M(S, \sigma) = M(y := y/sqrt(b[x]), \sigma) = \{\sigma[y \mapsto \beta]\}$ where $\beta = (\sigma(y)/sqrt(\gamma)) = (8/sqrt(\gamma))$ and $\gamma = \sigma(b)\ (\sigma(x)) = \sigma(b)(\alpha)$.
   So $\beta = \perp$ and thus $M(S, \sigma) = \{\sigma[y \mapsto \perp]\} = \{\perp_e\}$
   iff $\gamma = \perp$ or $\gamma < 0$ *or* $sqrt(\gamma) = 0$ ($b[x]$ fails, $b[x] < 0$, or $sqrt(b[x]]) = 0$)
   iff ($\alpha$ out of range for $\sigma(b)$] or $\gamma < 0$ or $sqrt(\gamma) = 0$ ($\gamma = \perp_e$ iff $b[x]$ has a bad index)
   iff ($\alpha < 0$ or $\alpha \geq 4$) or $\gamma = \sigma(b)(\alpha) < 0$ or $sqrt(\gamma) = 0$ ($\sigma(b)$ is of size 4)
   iff ($\alpha < 0$ or $\alpha \geq 4$) or ($\alpha = 0$) or $sqrt(\gamma) = 0$ (only $b[0] < 0$)
   iff ($\alpha < 0$ or $\alpha \geq 4$) or ($\alpha = 0$) or ($\alpha = 3$) (only $sqrt(b[3]) = sqrt(0) = 0$)
   iff ($\alpha \leq 0$ or $\geq 3$)
   (combining terms)

 