Binding and Substitution

$e ::= \dots \mid \text{let } x = e \text{ in } e \mid x$

$\text{let } x = e_1 \text{ in } e_2$ x is **bound** in e_2 (not e_1)

e.g. $\text{let } x = \bar{1} \text{ in } x + \bar{2}$.

If a variable isn't bound, it's free

FV(e) = Free variables of e

$FV(x) = \{x\}$

$FV(\bar{n}) = FV(\text{"s"}) = \emptyset$

$FV(\text{let } x = e_1 \text{ in } e_2) = FV(e_1) \cup \underbrace{(FV(e_2) \setminus \{x\})}_{x \text{ not free in } e_2}$

$FV(e_1 + e_2) = FV(e_1 \string^ e_2) = FV(e_1) \cup FV(e_2)$

$FV(e!) = FV(e)$

bound

? $\text{let } x = y \text{ in } x + \bar{2}$

free

bound

? $\text{let } x = x \text{ in } x + \bar{2}$

free

α conversion - Can always (consistently) rename **bound** vars

α - equivalent - expressions are the same up to α conversion

$\text{let } x = \bar{1} \text{ in } x + \bar{2} \equiv_\alpha \text{let } y = \bar{1} \text{ in } y + \bar{2}$

$\text{let } x = y \text{ in } x + \bar{2} \equiv_\alpha \text{let } z = y \text{ in } z + \bar{2}$

$\text{let } x = y \text{ in } x + \bar{2} \not\equiv_\alpha \text{let } x = z \text{ in } x + \bar{2}$

$\text{let } x = x \text{ in } x + \bar{2} \equiv_\alpha \text{let } y = x \text{ in } y + \bar{2}$

$\text{let } x = x \text{ in } x + \bar{2} \not\equiv_\alpha \text{let } x = y \text{ in } x + \bar{2}$

$\text{let } x = \bar{1} \text{ in } x + \bar{2}$

$\mapsto \bar{1} + \bar{2}$

$\mapsto \bar{3}$

Note: No state! <u>Substitute</u> $\bar{1}$ for x

$[e_1/x]\,e_2$ "substitute e_1 for <u>free instances of x</u> in e_2"

$[e/x]\,x = e$

$[e/x]\,y = y$ $\quad y \neq x$

$[e/x]\,\bar{n} = \bar{n}$

$[e/x]\,\text{"s"} = \text{"s"}$

$[e/x](e_1 + e_2) = [e/x]e_1 + [e/x]e_2$

$[e/x](e_1 \,\hat{}\, e_2) = [e/x]e_1 \,\hat{}\, [e/x]e_2$

$[e/x](\text{let } x = e_1 \text{ in } e_2) = \text{let } x = [e/x]e_1 \text{ in } e_2$

$[e/x](\text{let } y = e_1 \text{ in } e_2) = \text{let } y = [e/x]e_1 \text{ in } [e/x]e_2$ $\quad y \neq x,\ \underline{y \notin FV(e)}$

$[e/x](\text{let } x = \bar{1} \text{ in } x + \bar{2}) \neq \text{let } x = \bar{1} \text{ in } e + \bar{2}$

$\equiv_\alpha [e/x](\text{let } y = \bar{1} \text{ in } y + \bar{2}) = \text{let } y = \bar{1} \text{ in } y + \bar{2}$ $\cancel{\alpha}$

$[x + \bar{2}/y](\text{let } x = \bar{1} \text{ in } y + \bar{2}) \neq \text{let } x = \bar{1} \text{ in } x + \bar{2} + \bar{2}$

free!

now bound ("captured")

What if $y \in FV(e)$? α-convert

Dynamics "call-by-value"

$$\frac{e_1 \mapsto e_1'}{\text{let } x = e_1 \text{ in } e_2 \mapsto \text{let } x = e_1' \text{ in } e_2}\ (S-9) \qquad \frac{e_1 \text{ val}}{\text{let } x = e_1 \text{ in } e_2 \mapsto [e_1/x]\, e_2}$$

"call-by-name"

$$\frac{}{\text{let } x = e_1 \text{ in } e_2 \mapsto [e_1/x]\, e_2}\ (S-9)$$

Statics

Still have a context

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, x:\tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}\ (T-6)$$

Lemma (Substitution): If $\Gamma, x:\tau' \vdash e : \tau$ and $\Gamma \vdash e' : \tau'$, then $\Gamma \vdash [e'/x]e : \tau$.

Pf: By induction on the derivation of $\Gamma, x:\tau' \vdash e : \tau$.

$$\frac{}{\Gamma', y:\tau \vdash y : \tau}\ (T-0)$$ Case 1: $x = y$. Then $[e'/x]e = e'$ and $\tau = \tau'$. By assumption, $\Gamma \vdash e' : \tau'$.

Case 2: $x \neq y$. Then $[e'/x]e = y$. By T-0, $\Gamma \vdash y : \tau$.

$$\frac{}{\Gamma' \vdash \bar{n} : \text{int}}\ (T-1)$$ Then $[e'/x]e = \bar{n}$. Apply T-1.

$$\frac{\Gamma' \vdash e_1 : \text{int} \quad \Gamma' \vdash e_2 : \text{int}}{\Gamma' \vdash e_1 + e_2 : \text{int}}\ (T-3)$$ Then $[e'/x]e = [e'/x]e_1 + [e'/x]e_2$. By induction, $\Gamma \vdash [e'/x]e_1 : \text{int}$ and $\Gamma \vdash [e'/x]e_2 : \text{int}$. Apply (T-3).

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma, y:\tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } y = e_1 \text{ in } e_2 : \tau_2}\ (T-6)$$ Because of α-conv., we can assume $x \neq y$.

Then $[e'/x]e = \text{let } y = [e'/x]e_1 \text{ in } [e'/x]e_2$.

By IH, $\Gamma \vdash [e'/x]e_1 : \tau_1$.

We have $\Gamma, x:\tau', y:\tau_1 \vdash e_2 : \tau_2$. By weakening, $\Gamma, y:\tau_1 \vdash e' : \tau'$

(we can swap)

By induction, $\Gamma, y:\tau_1 \vdash [e'/x]e_2 : \tau_2$. Apply T-6. $\square$

At runtime, ctx is now empty

Preservation: If $\bullet \vdash e:\tau$ and $e \mapsto e'$ then $\bullet \vdash e':\tau$

$$\frac{(e_1 \text{ val})}{\text{let } x = e_1 \text{ in } e_2 \mapsto [e_1/x]e_2}$$

By inversion, $\bullet \vdash e_1 : \tau_1$ and $x:\tau_1 \vdash e_2 : \tau_2$.

By substitution, $\bullet \vdash [e_1/x]e_2 : \tau_2$. $\square$

Progress: If $\bullet \vdash e:\tau$ then e val or $e \mapsto e'$.

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x:\tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2}$$

CBN: Easy. Apply S-9.

CBV: By induction, e_1 val or $e_1 \mapsto e_1'$.

Apply S-10 Appy S-9. $\square$