Evaluation Contexts

(Yet another way to avoid search rules)

$e ::= x \mid () \mid \lambda x.e \mid e\ e \mid (e,e) \mid \text{fst } e \mid \text{snd } e$ $\quad v ::= () \mid \lambda x.e \mid (v,v)$

$\tau ::= \text{unit} \mid \tau \to \tau \mid \tau \times \tau$

$\mathcal{E} ::= \underset{[]}{\circ} \mid \mathcal{E}\ e \mid v\ \mathcal{E} \mid (\mathcal{E}, e) \mid (v, \mathcal{E}) \mid \text{fst } \mathcal{E} \mid \text{snd } \mathcal{E}$

Evaluation Contexts
- Exprs w/ one "hole"

$$\underbrace{(5, (6, (\lambda x.x)\ \underbrace{(\text{fst } (7,8))}_{\text{the part that can step}}))}_{\text{everything else}}$$

$\mathcal{E}[\text{fst } (7,8)]$

$\mathcal{E}[e]$ - fill the hole w/ e

$\circ[e] = e$

$(\mathcal{E}\ e)[e'] = (\mathcal{E}[e'])\ e$

$(v\ \mathcal{E})[e] = v\ \mathcal{E}[e]$

$(\mathcal{E}, e)[e'] = (\mathcal{E}[e'], e)$

$(v, \mathcal{E})[e'] = (v, \mathcal{E}[e'])$

$(\text{fst } \mathcal{E})[e] = \text{fst } (\mathcal{E}[e])$

$(\text{snd } \mathcal{E})[e] = \text{snd } (\mathcal{E}[e])$

$(5, (6, (\lambda x.x)\ \circ))[\text{fst } (7,8)]$
$= (5, (6, (\lambda x.x)\ (\text{fst } (7,8))))$

$$\frac{e \mapsto e'}{\mathcal{E}[e] \mapsto \mathcal{E}[e']}$$ - One big rule!

(still need to define this judgment)

$$\frac{}{(\lambda x.e)\ v \to [v/x]e} \qquad \frac{}{\text{fst } (v_1, v_2) \to v_1} \qquad \frac{}{\text{snd } (v_1, v_2) \to v_2}$$

Need "types for contexts"

$\mathcal{E} : \tau \rightsquigarrow \tau'$ Takes a τ (to fill the hole), acts as a τ'

No need for a context - $\mathcal{E}$s always represent closed exps

$$\frac{}{\circ : \tau \rightsquigarrow \tau}(1) \qquad \frac{\mathcal{E} : \tau \rightsquigarrow \tau_1 \to \tau_2 \quad \cdot \vdash e : \tau_1}{\mathcal{E}\ e : \tau \rightsquigarrow \tau_2}(2) \qquad \frac{\cdot \vdash v : \tau_1 \to \tau_2 \quad \mathcal{E} : \tau \rightsquigarrow \tau_1}{v\ \mathcal{E} : \tau \rightsquigarrow \tau_2}(3)$$

$$\frac{\mathcal{E} : \tau \rightsquigarrow \tau_1 \quad \cdot \vdash e : \tau_2}{(\mathcal{E}, e) : \tau \rightsquigarrow \tau_1 \times \tau_2}(4) \qquad \frac{\cdot \vdash v : \tau_1 \quad \mathcal{E} : \tau \rightsquigarrow \tau_2}{(v, \mathcal{E}) : \tau \rightsquigarrow \tau_1 \times \tau_2}(5) \qquad \frac{\mathcal{E} : \tau \rightsquigarrow \tau_1 \times \tau_2}{\mathsf{fst}\ \mathcal{E} : \tau \rightsquigarrow \tau_1}(6)$$

(snd) (τ_2) (7)

Lemma 1: If $\mathcal{E} : \tau \rightsquigarrow \tau'$ and $\cdot \vdash e : \tau$ then $\cdot \vdash \mathcal{E}[e] : \tau'$

Pf. By ind. on the derivation of $\mathcal{E} : \tau \rightsquigarrow \tau'$

(1) then $\mathcal{E} = \circ$ and $\mathcal{E}[e] = e$ and $\tau = \tau'$ ✓

(2) Then $\mathcal{E} = \mathcal{E}_0\ e_0$ and $\mathcal{E}_0 : \tau \rightsquigarrow \tau_1 \to \tau'$ and $\cdot \vdash e_0 : \tau_1$ and $\mathcal{E}[e] = \mathcal{E}_0[e]\ e_0$. By induction, $\cdot \vdash \mathcal{E}_0[e] : \tau_1 \to \tau'$. Apply $(\to E)$

(3) Then $\mathcal{E} = v\ \mathcal{E}_0$ and $\cdot \vdash v : \tau_1 \to \tau_2$ and $\mathcal{E}_0 : \tau \rightsquigarrow \tau_1$ and $\mathcal{E}[e] = v\ \mathcal{E}_0[e]$. By induction, $\cdot \vdash \mathcal{E}_0[e] : \tau_1$. Apply $(\to E)$

(4) Then $\mathcal{E} = (\mathcal{E}_0, e_0)$ and $\mathcal{E}_0 : \tau \rightsquigarrow \tau_1$ and $\cdot \vdash e_0 : \tau_2$, and $\mathcal{E}[e] = (\mathcal{E}_0[e], e_0)$ and $\tau' = \tau_1 \times \tau_2$. By ind., $\cdot \vdash \mathcal{E}_0[e] : \tau_1$. Apply $(\times I)$

(6) Then $\mathcal{E} = \mathsf{fst}\ \mathcal{E}_0$ and $\mathcal{E}_0 : \tau \rightsquigarrow \tau \times \tau_2$ and $\mathcal{E}[e] = \mathsf{fst}\ \mathcal{E}_0[e]$. By induction, $\cdot \vdash \mathcal{E}_0[e] : \tau_1 \times \tau_2$. Apply $(\times E_1)$ □

Lemma 2. If $\bullet \vdash \mathcal{E}[e] : \tau$ then there exists τ' such that $\bullet \vdash e : \tau'$ and $\mathcal{E} : \tau' \rightsquigarrow \tau$.

Proof. By induction on the structure of $\mathcal{E}$.

1. $\mathcal{E} = \circ$. Then $\mathcal{E}[e] = e$. Let $\tau' = \tau$. Apply (1)
2. $\mathcal{E} = \mathcal{E}_0\ e_0$. Then $\mathcal{E}[e] = \mathcal{E}_0[e]\ e_0$. By inversion on $\to E$, $\bullet \vdash \mathcal{E}_0[e] : \tau_1 \to \tau$ and $\bullet \vdash e_0 : \tau_1$. By induction, $\bullet \vdash e : \tau'$ and $\mathcal{E}_0 : \tau' \rightsquigarrow \tau_1 \to \tau$. Apply (2).
3. $\mathcal{E} = v\ \mathcal{E}_0$. Then $\mathcal{E}[e] = v\ \mathcal{E}_0[e]$. By inversion on $\to E$, $\bullet \vdash v : \tau_1 \to \tau$ and $\bullet \vdash \mathcal{E}_0[e] : \tau_1$. By induction, $\bullet \vdash e : \tau'$ and $\mathcal{E}_0 : \tau' \rightsquigarrow \tau_1$. Apply (3).
4. $\mathcal{E} = (\mathcal{E}_0, e_0)$. Then $\mathcal{E}[e] = (\mathcal{E}_0[e], e_0)$. By inversion on $\times I$, $\bullet \vdash \mathcal{E}_0[e] : \tau_1$ and $\bullet \vdash e_0 : \tau_2$ and $\tau = \tau_1 \times \tau_2$. By induction, $\bullet \vdash e : \tau'$ and $\mathcal{E}_0 : \tau' \rightsquigarrow \tau_1$. Apply (4)
6. $\mathcal{E} = \text{fst}\ \mathcal{E}_0$. Then $\mathcal{E}[e] = \text{fst}(\mathcal{E}_0[e])$. By inversion on $\times E_1$, $\bullet \vdash \mathcal{E}_0[e] : \tau \times \tau_2$. By induction, $\bullet \vdash e : \tau'$ and $\mathcal{E}_0 : \tau' \rightsquigarrow \tau \times \tau_2$. Apply (6).

Preservation: If $\bullet \vdash e : \tau$ and $e \mapsto e'$ then $\bullet \vdash e' : \tau$.

Case $\dfrac{e_0 \mapsto e_0'}{\mathcal{E}[e_0] \mapsto \mathcal{E}[e_0']}$ Then $e = \mathcal{E}[e_0]$. By Lemma 2, there exists τ' s.t. $\bullet \vdash e_0 : \tau'$ and $\mathcal{E} : \tau' \rightsquigarrow \tau$.
By induction, $\bullet \vdash e_0' : \tau'$. By Lemma 1, $\bullet \vdash \mathcal{E}[e_0'] : \tau$.

Other cases similar to past preservation proofs.

Lemma 3 (Decomposition). If $\bullet \vdash e : \tau$ then either e is a value or $e = \mathcal{E}[e']$ and e' is $(\lambda x.\tau'.e'')\ v$ or $\text{fst}\ (v_1, v_2)$ or $\text{snd}\ (v_1, v_2)$.

Pf. By induction on the derivation of $\bullet \vdash e : \tau$.

unit I, $\to I$. Then e is a value.

$\to_E$ Then $e = e_1\ e_2$ and $\bullet \vdash e_1 : \tau' \to \tau$ and $\bullet \vdash e_2 : \tau'$.

By induction, e_1 is a value or $e_1 = \mathcal{E}[e_1']$ and e_1' has one of the 3 forms.

e_1 value) By induction, e_2 is a value or $e_2 = \mathcal{E}[e_2']$ ...

e_2 is a value. Then by CF, $e = \circ[(\lambda x : \tau'. e'')\ v]$

$e_2 = \mathcal{E}[e_2']$. Then $e = v\ \mathcal{E}[e_2'] = (v\ \mathcal{E})[e_2']$ ...

Progress. If $\bullet \vdash e:\tau$ then e is a value or $e \mapsto e'$ for some e'.

Proof. By Lemma 3:

e is a value) ✓

$e = \mathcal{E}[e']$ and $e' = (\lambda x{:}\tau'.e'')\,v$) then $e \mapsto \mathcal{E}[[v/x]\,e'']$

$e' = \mathrm{fst}\ (v_1, v_2)$) Then $e \mapsto \mathcal{E}[v_1]$

$e' = \mathrm{snd}\ (v_1, v_2)$) Then $e \mapsto \mathcal{E}[v_2]$. □

Note there can be more than one way to decompose an exp

$$
\begin{aligned}
& (\lambda x{:}\mathrm{int}\times\mathrm{int}.\ \mathrm{fst}\ x)\ (\mathrm{fst}\ (\bar{1},\bar{2}),\bar{3}) \\
=\ & (\circ\ (\mathrm{fst}\ (\bar{1},\bar{2}),\bar{3}))[(\lambda x \ldots)] \\
=\ & ((\lambda x.\ \mathrm{fst}\ x)\ (\circ,\bar{3}))[\mathrm{fst}\ (\bar{1},\bar{2})] \quad \text{— the useful one} \\
=\ & ((\lambda x.\ \mathrm{fst}\ x)\ (\mathrm{fst}\ \circ,\bar{3}))[(\bar{1},\bar{2})] \\
\mapsto\ & (\lambda x{:}\mathrm{int}\times\mathrm{int}.\ \mathrm{fst}\ x)\ (\bar{1},\bar{3}) \\
=\ & ((\lambda x.\ \mathrm{fst}\ x)\ (\circ,\bar{3}))[\bar{1}] \quad \text{- no longer the useful one} \\
=\ & \circ[(\lambda x.\ \mathrm{fst}\ x)\ (\bar{1},\bar{3})] \\
\mapsto\ & \circ[\mathrm{fst}\ (\bar{1},\bar{3})] \\
\mapsto\ & \circ[\bar{1}]
\end{aligned}
$$

(Some parts of these notes derived from notes by David Walker @ Princeton)