Big Step Semantics

$e \mapsto e'$ "e steps to e'" (one step)
$e \mapsto^* e'$ "e steps to e'" (many steps)
$e \mapsto^* v$ and v val "e evaluates to v"

$e \Downarrow v$ "e evaluates to v"

$e ::= x \mid () \mid \lambda x. e \mid e\ e \mid (e, e) \mid \text{fst } e \mid \text{snd } e \mid \text{inl } e \mid$
$\quad \text{inr } e \mid \text{case } e \text{ of } \{x. e; y. e\} \mid \text{fix } x = e$
$\tau ::= \text{unit} \mid \tau \to \tau \mid \tau \times \tau \mid \tau + \tau$
$v ::= () \mid \lambda x. e \mid (v, v) \mid \text{inl } v \mid \text{inr } v$

Combines 3 step rules for application

$$\frac{}{v \Downarrow v} \text{(E-1)} \qquad \frac{e_1 \Downarrow \lambda x. e \quad e_2 \Downarrow v \quad [v/x]e \Downarrow v'}{e_1\ e_2 \Downarrow v'} \text{(E-2)}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)} \text{(E-3)} \qquad \frac{e \Downarrow (v_1, v_2)}{\text{fst } e \Downarrow v_1} \text{(E-4)} \qquad \frac{e \Downarrow (v_1, v_2)}{\text{snd } e \Downarrow v_2} \text{(E-5)}$$

$$\frac{e \Downarrow v}{\text{inl } e \Downarrow \text{inl } v} \text{(E-6)} \qquad \frac{e \Downarrow v}{\text{inr } e \Downarrow \text{inr } v} \text{(E-7)}$$

$$\frac{e_1 \Downarrow \text{inl } v \quad [v/x]e_2 \Downarrow v'}{\text{case } e_1 \text{ of } \{x. e_2; y. e_3\} \Downarrow v'} \text{(E-8)} \qquad \frac{e_1 \Downarrow \text{inr } v \quad [v/x]e_3 \Downarrow v'}{\text{case } e_1 \text{ of } \{x. e_2; y. e_3\} \Downarrow v'} \text{(E-9)}$$

Way less rules!

$$\frac{[\text{fix } x = e/x]e \Downarrow v}{\text{fix } x = e \Downarrow v} \text{(E-10)}$$

Preservation: If $\Gamma \vdash e : \tau$ and $e \Downarrow v$, then $\Gamma \vdash v : \tau$.

Pf

(E-1) ✓

(E-2) Then $e = e_1\, e_2$ and $e_1 \Downarrow \lambda x.e$ and $e_2 \Downarrow v'$ and $[v/x]e \Downarrow v$.

By inversion, $\tau = \tau_2$ and $\Gamma \vdash e_1 : \tau_1 \to \tau_2$ and $\Gamma \vdash e_2 : \tau_1$.

By induction, $\Gamma \vdash \lambda x.e : \tau_1 \to \tau_2$ and $\Gamma \vdash v' : \tau_1$.

By inversion, $\Gamma, x:\tau_1 \vdash e : \tau_2$.

By substitution, $\Gamma \vdash [v/x]e : \tau_2$. By induction, $\Gamma \vdash v : \tau_2$.

(E-3) Then $e = (e_1, e_2)$ and $e_1 \Downarrow v_1$ and $e_2 \Downarrow v_2$ and $v = (v_1, v_2)$.

By inversion, $\tau = \tau_1 \times \tau_2$ and $\Gamma \vdash e_1 : \tau_1$ and $\Gamma \vdash e_2 : \tau_2$.

By induction, $\Gamma \vdash v_1 : \tau_1$ and $\Gamma \vdash v_2 : \tau_2$.

By typing rules, $\Gamma \vdash v : \tau_1 \times \tau_2$.

(E-4) Then $e = \text{fst}\ e_0$ and $e_0 \Downarrow (v, v_2)$.

By inversion, $\Gamma \vdash e_0 : \tau \times \tau_2$. By induction, $\Gamma \vdash (v, v_2) : \tau \times \tau_2$.

By inversion, $\Gamma \vdash v : \tau$.

(E-6) Then $e = \text{inl}\ e_0$ and $e_0 \Downarrow v'$ and $v = \text{inl}\ v'$.

By inversion, $\tau = \tau_1 + \tau_2$ and $\Gamma \vdash e_0 : \tau_1$.

By induction, $\Gamma \vdash v' : \tau_1$. By typing rules, $\Gamma \vdash v : \tau_1 + \tau_2$.

(E-8) Then $e = \text{case}\ e_1\ \text{of}\ \{x.e_2 ; y.e_3\}$ and $e_1 \Downarrow \text{inl}\ v'$ and $[v'/x]e_2 \Downarrow v$.

By inversion, $\Gamma \vdash e_1 : \tau_1 + \tau_2$ and $\Gamma, x:\tau_1 \vdash e_2 : \tau$.

By induction, $\Gamma \vdash \text{inl}\ v' : \tau_1 + \tau_2$. By inversion, $\Gamma \vdash v' : \tau_1$.

By subst, $\Gamma \vdash [v'/x]e_2 : \tau$. By induction, $\Gamma \vdash v : \tau$.

(E-10) Then $e = \text{fix}\ x = e_0$ and $[\text{fix}\ x = e_0 / x]e_0 \Downarrow v$.

By inv, $\Gamma, x:\tau \vdash e : \tau$. By subst and ind, $\Gamma \vdash v : \tau$.

Progress: ...
One option: If $\Gamma \vdash e : \tau$, then $\exists v$ s.t. $e \Downarrow v$
True in STLC w/o fix but not in most real languages.

$\nexists v.$ fix $x = x \Downarrow v.$

Big-step can't talk about non-terminating expressions
No real way to talk about progress ($\Rightarrow$ type safety).

But: - Don't need to worry about evaluation order
- Don't need all the search rules

Thm. $e \mapsto^* v \iff e \Downarrow v$
$\Leftarrow$ Fairly straightforward with a couple annoying lemmas
$\Rightarrow$ Suffices to show
Lemma: If $e \mapsto e'$ and $e' \Downarrow v$ then $e \Downarrow v$.