Curry-Howard Isomorphism (Correspondence)

Intuitionistic Propositional Logic

$A, B ::= \top \mid \bot \mid A \wedge A \mid A \vee A \mid A \Rightarrow A$

Rules: $\dfrac{A \in \Gamma}{\Gamma \vdash A \text{ (true)}}$ $\quad \dfrac{}{\top}$ $\quad \dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}$ $\quad \dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$

$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$ $\quad \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$ $\quad \dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$ $\quad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$

$\dfrac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$ $\quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$ $\quad \dfrac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$

Modus Ponens: $(A \Rightarrow B) \wedge A \Rightarrow B$

$$\dfrac{\dfrac{\dfrac{\cdot, (A\Rightarrow B)\wedge A \vdash (A \Rightarrow B)\wedge A}{(A\Rightarrow B)\wedge A \vdash A \Rightarrow B} \quad \dfrac{\cdot, (A\Rightarrow B)\wedge A \vdash (A\Rightarrow B)\wedge A}{\cdot, (A\Rightarrow B)\wedge A \vdash A}}{\cdot, (A\Rightarrow B)\wedge A \vdash B}}{\cdot \vdash (A\Rightarrow B)\wedge A \Rightarrow B}$$

"Not": $\neg A \triangleq A \Rightarrow \bot$

In classical logic, $\neg\neg A \Leftrightarrow A$. Not true in IPL in general.
(and $A \vee \neg A$)

"law of the excluded middle"

But: $\neg\neg\neg A \Rightarrow \neg A$.

Compare IPL rules to STLC.

$\Gamma \vdash A$ (true) $\Leftrightarrow$ $\Gamma \vdash e : \tau$

Correspondence bt. props A + types τ: A true iff $\exists\, e,\ e:\tau$
So what are expressions?

$$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \wedge B}$$

e_1 = proof of A $\quad$ e_2 = proof of B
(e_1, e_2) = proof of $A \wedge B$!

Modus Ponens again: $(A \Rightarrow B) \wedge A \Rightarrow B$
As a type: $(A \to B) \times A \to B$

or $(\lambda x : ((A \to B) \times A).\ (\text{fst } x)\ (\text{snd } x)) : (A \to B) \times A \to B$

Recall: $\neg\neg\neg A \Rightarrow \neg A$ $\quad$ As type: $(((A \to \text{void}) \to \text{void}) \to \text{void}) \to (A \to \text{void})$

$\lambda x : ((A \to \text{void}) \to \text{void}) \to \text{void}.$
$\quad \lambda y : A.\ x\ (\lambda z : (A \to \text{void}).\ z\ y)$

De Morgan: $(\neg(A \vee B) \Rightarrow \neg A \wedge \neg B) \wedge (\neg A \wedge \neg B \Rightarrow \neg(A \vee B))$

$(\ \lambda x : ((A \vee B) \to \text{void}).\ (\lambda a : A.\ x\ (\text{inl } a),\ \lambda b : B.\ x\ (\text{inr } b)),$
$\quad \lambda x : (A \to \text{void}) \times (B \to \text{void}).\ \lambda y : A \vee B.$
$\qquad \text{case } y\ \{ a.(\text{fst } x)\ a\ ;\ b.(\text{snd } x)\ b \}\ )$