Weakest Preconditions

Part 2: Calculating wp, wlp; Domain Predicates

CS 536: Science of Programming, Spring 2023

A. Why

- The weakest precondition and weakest liberal preconditions are the most general preconditions that a program needs in order to run correctly.

B. Objectives

At the end of this activity you should be able to

- Describe the relationship between $wp(S, q_1 \vee q_2)$, $wp(S, q_1)$, and $wp(S, q_2)$ and how it differs for deterministic and nondeterministic programs.
- Be able to calculate the *wlp* of a simple loop-free program.

C. Problems

1. How are $wp(S, q_1 \vee q_2)$ and $wp(S, q_1)$ and $wp(S, q_2)$, related if S is deterministic? If S is nondeterministic?

For Problems 2 – 4, just syntactically calculate the *wlp*; don't also logically simplify the result.

2. Calculate the *wlp* in each of the following cases.
   a. $wlp(k := k - s, n = 3 \wedge k = 4 \wedge s = -7)$.
   b. $wlp(n := n*(n-k), n = 3 \wedge k = 4 \wedge s = -7)$.
   c. $wlp(n := n*(n-k)\ ;\ k := k - s, n > k + s)$
3. Let $Q(k, s) \equiv 0 \le k \le n \wedge s = sum(0, k)$ where $sum(u, v)$ is the sum of u, $u+1$, …, v (when $u \le v$) or 0 (when $u > v$).
   a. Calculate $wp(k := k+1;\ s := s+k\ , Q(k, s))$.
   b. Calculate $wp(s := s+k+1;\ k := k+1, Q(k, s))$.
   c. Calculate $wp(s := s+k;\ k := k+1, Q(k, s))$. (This one isn't compatible with $s = sum(0, k)$.)

For Problems 4 – 6, don't forget the domain predicates. You can logically simplify as you go.

4. Calculate the *wp* below.
   a. $wp($**if** B **then** $x := x/2$ **fi**$;\ y := x, x = 5 \wedge y = z)$.
   b. $wp($**if** $x \ge 0$ **then** $x := x*2$ **else** $x := y$ **fi**$;\ x := c*x, a \le x < y)$
5. Calculate p to be the *wp* in $\{p\}\ x := y/b[k]\ \{x > 0\}$.
6. Calculate p_1 and p_2 to be the *wp's* in $\{p_1\}\ y := sqrt(b[k])\ \{z < y\}$ and $\{p_2\}\ k := x/k\ \{p_1\}$.

Solution to Practice 11 (Weakest Preconditions, pt. 2)

1. For deterministic S, $wp(S, q_1 \vee q_2) \Leftrightarrow wp(S, q_1) \vee wp(S, q_2)$.
   For nondeterministic S, we have $\Rightarrow$ but not $\Leftarrow$.

2. (Calculate *wlp*)
   a. $wlp(k := k - s, n = 3 \wedge k = 4 \wedge s = -7) \equiv n = 3 \wedge k-s = 4 \wedge s = -7$
   b. $wlp(n := n*(n-k), n = 3 \wedge k = 4 \wedge s = -7) \equiv n*(n-k) = 3 \wedge k = 4 \wedge s = -7$
   c. $wlp(n := n*(n-k); k := k-s, n > k+s)$
      $\equiv wlp(n := n*(n-k), wlp(k := k-s, n > k+s))$
      $\equiv wlp(n := n*(n-k), n > k-s+s)$
      $\equiv n*(n-k) > k-s+s$

3. (*wp* involving sums) We have $Q(k, s) \equiv 0 \le k \le n \wedge s = sum(0, k)$.
   a. $wp(k := k+1; s := s+k, Q(k, s))$
      $\equiv wp(k := k+1, wp(s := s+k, Q(k, s))$
      $\equiv wp(k := k+1, Q(k, s+k))$
      $\equiv wp(k := k+1, 0 \le k \le n \wedge s+k = sum(0, k))$
      $\equiv 0 \le k+1 \le n \wedge s+k+1 = sum(0, k+1)$

   b. $wp(s := s+k+1; k := k+1, Q(k, s))$
      $\equiv wp(s := s+k+1, wp(k := k+1, Q(k, s))$
      $\equiv wp(s := s+k+1, Q(k+1, s))$
      $\equiv wp(s := s+k+1, 0 \le k+1 \le n \wedge s = sum(0, k+1))$
      $\equiv 0 \le k+1 \le n \wedge s+k+1 = sum(0, k+1)$

   c. $wp(s := s+k; k := k+1, Q(k, s))$
      $\equiv wp(s := s+k, wp(k := k+1, Q(k, s))$
      $\equiv wp(s := s+k, Q(k+1, s))$
      $\equiv wp(s := s+k, 0 \le k+1 \le n \wedge s = sum(0, k+1))$
      $\equiv 0 \le k+1 \le n \wedge s+k = sum(0, k+1)$. Note this isn't compatible with $s = sum(0, k)$.

4. (*wp* of ***if-then***)
   a. *wp(**if** B **then** x := x/2 **fi**; y := x, x = 5 ∧ y = z)*
      *≡ wp(**if** B **then** x := x/2 **fi**, wp(y := x, x = 5 ∧ y = z))*
      *≡ wp(**if** B **then** x := x/2 **fi**, x = 5 ∧ x = z)*
      *≡ (B → wp(x := x/2, x = 5 ∧ x = z)) ∧ (¬B → wp(**skip**, x = 5 ∧ x = z))*
      $\equiv (B \rightarrow x/2 = 5 \wedge x/2 = z) \wedge (\neg B \rightarrow x = 5 \wedge x = z)$

b. $wp(\textbf{if } x \ge 0 \textbf{ then } x := x*2 \textbf{ else } x := y \textbf{ fi};\ x := c*x,\ a \le x < y)$.

$\equiv wp(S, wp(x := c*x, a \le x < y))$ where S is the ***if*** statement

$\equiv wp(S, a \le c*x < y)$

$\equiv wp(\textbf{if } x \ge 0 \textbf{ then } x := x*2 \textbf{ else } x := y \textbf{ fi}, a \le c*x < y)$

$\equiv (x \ge 0 \rightarrow wp(x := x*2, a \le c*x < y)) \wedge (x < 0 \rightarrow wp(x := y, a \le c*x < y))$

$\equiv (x \ge 0 \rightarrow a \le c*(x*2) < y) \wedge (x < 0 \rightarrow a \le c*y < y)$

5. For $\{p\}\ x := y/b[k]\ \{x > 0\}$,

let $p \Leftrightarrow wp(x := y/b[k], x > 0)$

$\equiv wlp(x := y/b[k], x > 0) \wedge D(x := y/b[k])$

$\equiv y/b[k] > 0 \wedge b[k] \ne 0 \wedge D(b[k])$

$\equiv y/b[k] > 0 \wedge b[k] \ne 0 \wedge 0 \le k < size(b)$

6. For $\{p_1\}\ y := sqrt(b[k])\ \{z < y\}$

let $p_1 \Leftrightarrow wp(y := sqrt(b[k]), z < y)$

$\equiv wlp(y := sqrt(b[k]), z < y) \wedge D(y := sqrt(b[k]))$

$\equiv z < sqrt(b[k]) \wedge b[k] \ge 0 \wedge D(b[k])$

$\equiv z < sqrt(b[k]) \wedge b[k] \ge 0 \wedge 0 \le D(b[k] < size(b)$

For $\{p_2\}\ k := x/k;\ \{p_1\}$, let

$p_2 \Leftrightarrow wp(k := x/k, p_1)$

$\equiv wlp(k := x/k, p_1) \wedge D(k := x/k)$

$\equiv p_1[x/k\ /\ k] \wedge k \ne 0$

$\equiv (z < sqrt(b[k]) \wedge b[k] \ge 0 \wedge 0 \le D(b[k] < size(b))\ [x/k\ /\ k] \wedge k \ne 0$

$\equiv z < sqrt(b[x/k]) \wedge b[x/k] \ge 0 \wedge 0 \le D(b[x/k] < size(b) \wedge k \ne 0$

