Denotational Semantics; Runtime Errors

CS 536: Science of Programming, Spring 2023

A. Why

- Our simple programming language is a model for the kind of constructs seen in actual languages.
- Our programs stand for state transformers.
- Runtime errors cause failure of normal program execution.

B. Outcomes

At the end of today, you should be able to:

- Give the denotational semantics of a program in a state.
- Say when and how evaluation of an expression or program fails due to a runtime error.

C. Problems

Denotational Semantics

Problems 1 – 4 are the denotational versions of the similar questions from Practice 5.

1. What is
   a. $M(x := x+1, \{x = 5\})$?
   b. $M(x := x+1, \sigma)$? (Your answer will be symbolic.)
   c. $M(x := x+1;\ y := 2*x, \{x = 5\})$?

2. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x+1$ ***else*** $y := 2*x$ ***fi***.
   a. Let $\sigma(x) = 8$. What is $M(IF, \sigma)$?
   b. Repeat, if $\sigma(x) = 0$.
   c. Repeat, if we don't know what $\sigma(x)$ is. (Your answer will be symbolic and have cases.)

3. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x/z$ ***fi***.
   a. What is $M(IF, \sigma)$ if $\sigma = \{x = 8, z = 3\}$? (Don't forget, integer division truncates)
   b. What is $M(IF, \{x = -2, z = 3\})$?

4. Let $W \equiv$ ***while*** $x < 3$ ***do*** S ***od*** where $S \equiv x := x+1;\ y := y*x$.
   a. Evaluate the body S in an arbitrary state τ and give $M(S, \tau)$.
   b. What is $M(W, \sigma)$ if $\sigma \vDash x = 4 \wedge y = 1$?
   c. What is $M(W, \sigma)$ if where $\sigma \vDash x = 1 \wedge y = 1$?

5. Let W be the program from question 4.
   a. What (if any) are the states such that $M(W, \sigma)=\{\perp\}$?
   b. Let V be the program W except that it uses $x = 3$ instead of $x < 3$. What (if any) are the states such that $M(V, \sigma)=\{\perp\}$?

Runtime Errors

For Problems 6 and 7, remember that we're using integer division and square root that truncate toward zero. E.g., $2 / 3 = 0$, $4 / 3 = 1$, $sqrt(3) = 1$, $sqrt(8) = 2$, *sqrt* of $(15) = 3$, etc.

6. Let $S \equiv x := y / b[x]$ and let $\sigma = \{b = (3, 0, -2, 4), x = \beta, y = 13\}$. Find all σ such that $M(S, \sigma) = \{\perp_e\}$.

7. Repeat the previous problem on $S \equiv y := y / sqrt(b[x])$ and $\sigma = \{b = (0, 9, 12, -3, 4), x = \beta, y = 2\}$

8. What are the results of replacing σ below by ⊥? (This is for arbitrary S, S_1, S_2, β, and τ.)
   a. $M(S, \sigma)$
   b. $\sigma[x \mapsto \beta]$
   c. $M(S_1; S_2, \tau) = M(S_2, \sigma) = \tau_1$, where $\sigma = M(S_1, \tau)$.
   d. $\sigma \vDash 2 < 3$
   e. $\sigma \nvDash 3 < 1$

Solution to Practice 6 (Denotational Semantics; Runtime Errors)

Denotational Semantics

1. (Calculate meanings of programs)
   a. $M(x := x+1, \{x = 5\}) = \{\{x = 5\}[x \mapsto \{x = 5\}(x+1)]\} = \{\{x = 6\}\}$
   b. $M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto \sigma(x)+1]\}$
   c. $M(x := x+1;\ y := 2*x, \{x = 5\})$
      $= M(y := 2*x,\ M(x := x+1, \{x = 5\})$
      $= M(y := 2*x,\ \{x = 6\})$, from part (a)
      $= \{\{x = 6\}[y \mapsto \beta]\}$ where $\beta = \{x = 6\}(2*x) = 12$
      $= \{\{x = 6,\ y = 12\}\}$

2. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x+1$ ***else*** $y := 2*x$ *fi*.
   a. If $\sigma(x) = 8$, then $\sigma(x > 0) = T$, so $M(IF, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto 9]\}$
   b. If $\sigma(x) = 0$, then $\sigma(x > 0) = F$, so $M(IF, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto \sigma(2*x)]\} = \{\sigma[y \mapsto 0]\}$
   c. If $\sigma(x) > 0$ then $M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\}$
      If $\sigma(x) \le 0$ then $M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto 2 \times \sigma(x)]\}$

3. Let $IF \equiv$ *if* $x > 0$ ***then*** $x := x/z$ ***fi*** $\equiv$ ***if*** $x > 0$ ***then*** $x := x/z$ ***else skip fi***
   a. If $\sigma = \{x = 8, z = 3\}$, then $\sigma(x > 0) = T$, so $M(IF, \sigma) = M(x := x/z, \sigma) = \{\sigma[x \mapsto \beta]\}$ where $\beta = \sigma(x/z)$ $= \sigma[x \mapsto 8 \div 3] = \sigma[x \mapsto 2]$, since integer division truncates.
   b. If $\sigma = \{x = -2, z = 3\}$ then $\sigma(x > 0) = F$, so $M(IF, \sigma) = M($***skip***$, \sigma) = \{\sigma\}$.

4. Let $W \equiv$ ***while*** $x < 3$ ***do*** S ***od*** where $S \equiv x := x+1;\ y := y*x$.
   a. For arbitrary τ,
      $M(S, \tau) = M(x := x+1;\ y := y*x, \tau)$
      $= M(y := y*x, \tau[x \mapsto \tau(x)+1])$
      $= \{\ \tau[x \mapsto \tau(x)+1][y \mapsto \beta]\ \}$ where $\beta = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1)$
   b. If $\sigma \vDash x = 4 \wedge y = 1$, then $\sigma(x < 3) = F$ so $M(W, \sigma) = \{\sigma\}$.
   c. If $\sigma \vDash x = 1 \wedge y = 1$, then $\sigma(x < 3) = T$ so we have at least one iteration to do.
      Let $\sigma_0 = \sigma$, let $\sigma_1 = M(S, \sigma_0) = \sigma_0(y) \times (\sigma_0(x)+1)$, and let $\sigma_2 = M(S, \sigma_1) = \sigma_1(y) \times (\sigma_1(x)+1)$.
      Then,
      $\sigma_0 = \sigma[x \mapsto 1][y \mapsto 1]$
      $\sigma_1 = M(S, \sigma_0) = \sigma_0[x \mapsto \sigma_0(x)+1][y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2]$
      $\sigma_2 = M(S, \sigma_1) = \sigma_1[x \mapsto 2+1][y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6]$
      Since σ_0 and $\sigma_1 \vDash x < 3$ but $\sigma_2 \vDash x \ge 3$, we have $M(W, \sigma) = \{\sigma_2\} = \{\sigma[x \mapsto 3][y \mapsto 6]\}$.

5. We have $W \equiv$ ***while*** *x < 3* ***do*** *S* ***od*** where $S \equiv x := x+1;\ y := y*x$.
   a. There are no states such that $M(W, \sigma)=\{\perp\}$. If $\sigma(x) \geq 3$, then W halts immediately and $M(W, \sigma) = \{\sigma\}$. If $\sigma(x) < 3$, then W will run until x=3, modifying y as appropriate.
   b. Let $W \equiv$ ***while*** *x = 3* ***do*** *S* ***od*** where $S \equiv x := x+1;\ y := y*x$. The states that cause $M(W, \sigma) = \{\perp\}$ are the ones in which $\sigma(x) > 3$; since x only increases, it can never become 3. On the other hand, if $\sigma(x) \leq 3$, then W will do 3-x iterations and halt when x = 3.

Runtime Errors

6. We have $S \equiv x := y\ /\ b[x]$ and $\sigma = \{b = (3, 0, -2, 4), x = \beta, y = 13\}$, and we want all σ such that $M(S, \sigma) = \{\perp_e\}$.
   $M(S, \sigma) = M(x := y/b[x], \sigma) = \{\sigma[x \mapsto \delta]\}$ *where* $\delta = \sigma(y/b[x]) = 13/\sigma(b)(\beta)$. To get $M(S, \sigma) = \{\perp_e\}$, then, we need $\delta = \perp_e$.
   So $\delta = \perp_e$
   iff $\sigma(b)(\beta) = \perp_e$ or $\sigma(b)(\beta) = 0$
   iff (β is out of range for $\sigma(b)$) or $(\sigma(b)(\beta) = 0)$ ($b[x]$ fails if x is out of range)
   iff $(\beta < 0$ *or* $\beta \geq 4)$ or $(\sigma(b)(\beta) = 0)$ ($\sigma(b)$ has size 4)
   iff $(\beta < 0$ *or* $\beta \geq 4)$ or $(\beta = 1)$ ($b[1]$ is the only element = 0)
   iff $\neg(\beta = 0, 2,$ or $3)$

7. Repeat, with $S \equiv y := y\ /\ sqrt(b[x])$ and $\sigma = \{b = (0, 9, 12, -3, 4), x = \beta, y = 2\}$.
   We have $M(S, \sigma) = M(y := y/sqrt(b[x]), \sigma) = \{\sigma[y \mapsto \beta]\}$ where $\beta = \sigma(y/sqrt(b[x]))$. Then,
   $M(S, \sigma) = \{\perp_e\}$
   iff $\sigma[y \mapsto \beta] = \perp_e$
   iff $\beta = \perp_e$ (since $\sigma \neq \perp$)
   iff $\beta = \sigma(y/sqrt(b[x])) = \sigma(y)\ /\ sqrt(\ \sigma(b)(\sigma(b[x]))) = \perp_e$
   iff $\sigma(b)(\sigma(b[x])) = \perp_e$ or $\sigma(b)(\sigma(b[x])) \leq 0$ (to get sqrt(negative number) or division by 0)
   iff σ(b[x]) is out of range for b, or σ(x) = 0 or 3 (since b[0] = 0 and b[3] < 0)
   iff σ(x) < 0 or > 4 or is 0 or is 3
   iff σ(x) ≤ 0 or = 3 or >4

8. (Using ⊥ as a state) Replacing σ with ⊥,
   a. $M(S, \perp) = \{\perp\}$
   b. $\perp[x \mapsto \beta] = \perp$
   c. $M(S_1; S_2, \tau_0) = M(S_2, \perp) = \perp$
   d. $\perp \not\vDash 2 < 3$
   e. $\perp \not\vDash 3 < 1$

