Satisfaction, Validity, and State Updates

CS 536: Science of Programming, Fall 2023

A. Why

- A predicate is satisfied or unsatisfied relative to a state.
- A predicate is valid if it is satisfied in all states.
- State updates occur when we introduce new variables or change the values of existing variables.

B. Outcomes

At the end of today, you should

- Know how to check a predicate for satisfaction in a state, how to check a predicate for validity, and know how to update a state.

C. Questions

1. Say u and v stand for variables (possibly the same variable) and α and β are values (possibly equal). When is $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$? Hint: There are four cases because maybe $u \equiv v$ and maybe $\alpha = \beta$.

2. Let $\sigma(b) = (7, 5, 12, 16)$. Assume out-of-bound indexes cause runtime errors.
   a. Does $\sigma \models \exists\, k\,.\, 0 \le k \wedge k{+}1 < size(b) \wedge b[k] < b[k{+}1]$? If so, what was your witness value for *k*?
   b. Does $\sigma \models \exists\, k\,.\, 0 \le k{-}1 \wedge k{+}1 < size(b) \wedge b[k{-}1] < b[k] < b[k{+}1]$? If so, what was your witness value for *k*?
   c. Does $\sigma \models \forall\, k\,.\, 0 \le k < 4 \rightarrow b[k] > 0$?
   d. If $\sigma(k) = -5$, then does $\sigma \models \exists\, k\,.\, 0 \le k < 4 \wedge b[k] > 0$?

3. For each of the situations below, fill in the blanks to describe when the situation holds.

   Fill in $\_\_\_\_\__{1}$ with "*some*", "*every*", or "*this*"

   Fill in $\_\_\_\_\__{2}$ with "*some*" or "*every*"

   Fill in $\_\_\_\_\__{3}$ with "σ(*x*) must *be undefined*", "σ(*x*) must *be defined and* $\sigma \models p$", or "*nothing of* σ(*x*)"

   Fill in $\_\_\_\_\__{4}$ with "$\models p$" or "$\not\models p$"

   a. $\sigma \models (\exists\, x \in U.\ p)$ iff for $\_\_\_\_\__{1}$ state σ and $\_\_\_\_\__{2}$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ $\_\_\_\_\__{4}$

b. $\sigma \vDash (\forall\, x \in U.\ p)$ iff for ______ $_1$ state σ and ______ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ ______ $_4$

c. $\sigma \vDash (\exists\, x \in U.\ p)$ requires ______ $_3$.

d. $\sigma \vDash (\forall\, x \in U.\ p)$ requires ______ $_3$.

e. $\sigma \nvDash (\exists\, x \in U.\ p)$ iff for ______ $_1$ state σ for ______ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ ______ $_4$

f. $\sigma \nvDash (\forall\, x \in U.\ p)$ iff for ______ $_1$ state σ for ______ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ ______ $_4$

g. $\nvDash (\forall\, x \in U.\ p)$ iff for ______ $_2$ state σ, we have σ ______ $_4$ $(\forall\, x \in U.\ p)$.

h. $\nvDash (\exists\, x \in U.\ p)$ iff for ______ $_2$ state σ, we have σ ______ $_4$ $(\exists\, x \in U.\ p)$.

i. $\nvDash (\forall\, x \in U.\ p)$ iff for ______ $_2$ state σ, and for ______ $_2$ $\alpha \in U$, we have $\sigma[x \mapsto \alpha]$ ______ $_4$

j. $\vDash (\exists\, x \in U\,.\ (\forall\, y \in V\,.\ p))$ iff for ______ $_1$ state σ, for ______ $_2$ $\alpha \in U$, and for ______ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ ______ $_4$

k. $\nvDash (\exists\, x \in U\,.\ (\forall\, y \in V\,.\ p))$ iff for ______ $_1$ state σ, for ______ $_2$ $\alpha \in U$, and for ______ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ $[\vDash \mid \vDash \neg]$ p.

l. $\vDash (\forall\, x \in U\,.\ (\exists\, y \in V\,.\ p))$ iff for ______ $_1$ state σ, for ______ $_2$ $\alpha \in U$, and for ______ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ $[\vDash \mid \vDash \neg]$ p.

m. $\nvDash (\forall\, x \in U\,.\ (\exists\, y \in V\,.\ p))$ iff for ______ $_1$ state σ, for ______ $_2$ $\alpha \in U$, and for ______ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ ______ $_4$

n. $\sigma \nvDash \exists\, x \in U\,.\ (\exists\, y \in V\,.\ p(x, y)) \rightarrow (\exists\, z \in W\,.\ q(x, z))$ iff for ______ $_1$ state σ, for ______ $_2$ $\alpha \in U$, if for ______ $_2$ $\beta \in V$, $\sigma[x \mapsto \alpha][y \mapsto \beta]$ ______ $_4$ $p(x, y)$, then for ______ $_2$ $\delta \in W$, $\sigma[x \mapsto \alpha][[z \mapsto \delta]$ ______ $_4$ $q(x, z)$.

4. Let $p \equiv \exists\, y\,.\ \forall\, x\,.\ f(x) > y$, and let $q \equiv \forall\, x\,.\ \exists\, y\,.\ f(x) > y$. (As usual, assume a domain of $\mathbb{Z}$.)

   a. Is it the case that for any *f*, if *p* is valid then so is *q*? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash p$ but $\nvDash q$.

   b. (The converse.) Is it the case that for any *f*, if *q* is valid then so is *p*? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash q$ but $\nvDash p$.

CS 536: Solution to Activity 4 (Satisfaction, Validity, and State Updates)

1. $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$ iff $u \not\equiv v$ or $\alpha = \beta$, or more precisely, iff $u \not\equiv v$ or ($u \equiv v$ and) $\alpha = \beta$.

2. (Quantified statements over arrays) Let $\sigma(b) = (7, 5, 12, 16)$.
   a. Yes, $\sigma \vDash \exists\, k\,.\, 0 \le k \wedge k+1 < size(b) \wedge b[k] < b[k+1]$ with 1 and 2 as possible witnesses for k.
   b. Yes, $\sigma \vDash \exists\, k\,.\, 0 \le k\text{-}1 \wedge k+1 < size(b) \wedge\ b[k\text{-}1] < b[k] < b[k+1]$ with 2 as the only witness that works.
   c. Yes, $\sigma \vDash \forall\, k\,.\, 0 \le k < 4 \rightarrow b[k] > 0$, since $b[0]$, $b[1]$, $b[2]$, and $b[3]$ are all positive in σ. Recall we're looking for an α such that $\sigma[k \mapsto \alpha] \vDash 0 \le k < 4 \rightarrow b[k] > 0$, and for $\sigma[k \mapsto \alpha]$, it doesn't matter whether $\sigma(k)$ has a value or what that value is.

      No: $\sigma \nvDash \forall\, k\,.\, 0 \le k < 4 \wedge b[k] > 0$ because there are plenty of values for k that are not in the range 0 through 3. (So whether the body uses → or ∧ is extremely important.)
   d. Yes, $\sigma \vDash \exists\, k\,.\, 0 \le k < 4 \wedge b[k] > 0$, with witnesses k = 0, 1, 2, or 3. (Again, σ(k) is irrelevant.)

      Yes (and perhaps surprisingly), $\sigma \vDash \exists\, k\,.\, 0 \le k < 3 \rightarrow b[k] < 0$ with witness $k = 3$:
      $\sigma[k \mapsto 3]$ satisfies $0 \le k < 3 \rightarrow b[k] < 0$ because 3 makes $0 \le k < 3$ false, so the implication is true even though the value of $b[3]$ is positive. (I'm avoiding k outside the range of b because those $b[k]$ cause runtime errors.)

3. (Validity/invalidity of quantified predicates)
   a. this σ, some α, $\vDash p$
   b. this σ, every α, $\vDash p$
   c. nothing of $\sigma(x)$
   d. nothing of $\sigma(x)$
   e. this σ, every α, $\nvDash p$
   f. this σ, some α, $\nvDash p$
   g. some σ, $\nvDash$
   h. some σ, $\nvDash$
   i. some σ, some α, $\nvDash p$
   j. every σ, some α, every β, $\vDash p$
   k. some σ, every α, some β, $\nvDash p$
   l. every σ, every α, some β, $\vDash p$
   m. some σ, some α, every β, $\nvDash p$
   n. this σ, every α, some β, $\vDash q$, every δ, $\nvDash p$ because the negation of *$(\exists\, x\,.\, ((\exists\, y \ldots) \rightarrow (\exists\, z \ldots)))$* is *$(\forall\, x\,.\, ((\exists\, y \ldots) \wedge \neg(\exists\, z \ldots)))$*.

4. ($\exists\forall$ predicates versus $\forall\exists$ predicates, specifically $p \equiv \exists y\,.\,\forall x\,.\,f(x) > y$, and $q \equiv \forall x\,.\,\exists y\,.\,f(x) > y$)

   a. The relation does hold: $\models p$ implies $\models q$. The short explanation is that for satisfaction of q, for each value α for x, we need to find a value β for y that satisfies the body $f(x) > y$. Now, p says that there's a value that works for every α, so we can use that value for β.

      In more detail, assume p is valid: for every state σ, there is some value β where for every value α, $\sigma[y \mapsto \beta][x \mapsto \alpha] \models f(x) > y$.

      To show that q is valid, take an arbitrary state τ with value δ for x. We need a witness value for the $\exists y$; since $\tau \models p$, there's a β for the $\exists y$ of p, and we'll use that as the witness for the $\exists y$ in q. To satisfy q, we need $\tau[x \mapsto \delta][y \mapsto \beta] \models f(x) > y$. Since $x \not\equiv y$, it doesn't matter whether we update using x and then y or vice versa. So it's sufficient to know $\tau[y \mapsto \beta][x \mapsto \delta] \models f(x) > y$, and we know that from $\tau \models p$.

   b. The relation does not hold: We can have $\models q$ but $\not\models p$. An easy example is $f(x) = x$, then validity of p would require us to find a value in $\mathbb{Z}$ for y that is > every value of x in $\mathbb{Z}$, but no such value exists.

      As an aside, if use an arbitrary predicate over x and y as the body of the $\exists\forall$ and $\forall\exists$ predicates, then the relation holds for some predicates and not for others. For example, $\exists x.\forall y\,.\,x \le y^2$ and $\forall y.\exists x.\,x \le y^2$ both hold.