Denotational Semantics; Runtime Errors

CS 536: Science of Programming, Fall 2021

A. Why

• Our simple programming language is a model for the kind of constructs seen in actual languages.
• Our programs stand for state transformers.
• Runtime errors cause failure of normal program execution.

B. Outcomes

At the end of today, you should be able to

• Give the denotational semantics of a program in a state.
• Say when and how evaluation of an expression or program fails due to a runtime error.

C. Problems

Denotational Semantics

Problems 1 – 4 are the denotational versions of the similar questions from Practice 5

1. What is
   a. \( M(x := x+1, \{x = 5\}) \)?
   b. \( M(x := x+1, \sigma) \)? (Your answer will be symbolic.)
   c. \( \langle x := x+1; y := 2^*x, \{x = 5\} \rangle \)?

2. Let \( S = \text{if } x > 0 \text{ then } x := x+1 \text{ else } y := 2^*x \text{ fi.} \)
   a. Let \( \sigma(x) = 8 \). What is \( M(S, \sigma) \)?
   b. Repeat, if \( \sigma(x) = 0 \).
   c. Repeat, if we don't know what \( \sigma(x) \) is. (Your answer will be symbolic.)

3. Let \( S = \text{if } x > 0 \text{ then } x := x/z \text{ fi.} \)
   a. What is \( M(S, \sigma) \) if \( \sigma = \{x = 8, z = 3\} \)? (Don't forget, integer division truncates)
   b. What is \( M(S, \{x = -2, z = 3\}) \)?

4. Let \( W = \text{while } x < 3 \text{ do } S \text{ od} \) where \( S = x := x+1; y := y^*x \).
   a. Evaluate the body \( S \) in an arbitrary state \( \tau \) and give \( M(S, \tau) \).
   b. What is \( M(W, \sigma) \) if \( \sigma = x = 4 \land y = 1 \)?
   c. What is \( M(W, \sigma) \) if where \( \sigma = x = 1 \land y = 1 \)?
Runtime Errors

5. Let $S = x := y/b[x]$ and let $\sigma = \{b = (3, 0, -2, 4), x = \alpha, y = 13\}$. Find all $\alpha$ such that $M(S, \sigma) = \{\bot_e\}$. (Remember, integer division truncates.)

6. Repeat the previous problem on $S = y := y / \sqrt{b[x]}$ and $\sigma = \{b = (-1, 9, 12, 0), x = \alpha, y = 8\}$. Treat $\sqrt{}$ as returning the truncated integer square root of its argument. (I.e., $\sqrt{0} = 0$, $\sqrt{1}$ through $3$ are all $1$, $\sqrt{4}$ through $8 = 2$, etc.)
Solution to Practice 6 (Denotational Semantics; Runtime Errors)

Denotational Semantics

1. (Calculate meanings of programs)
   a. \( M(x := x+1, \{x = 5\}) = \{\{x=5\}[x \mapsto \{x = 5\}(x+1)]\} = \{\{x = 6\}\} \)
   b. \( M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto \sigma(x)+1]\} \)
   c. \( M(x := x+1; y := 2*x, \{x = 5\}) \)
      \[ M(y := 2*x, M(x := x+1, \{x = 5\}) \]
      \[ = M(y := 2*x, M(x := x+1, \{x = 5\}) \]
      \[ = \{\{x=6\}[y \mapsto \beta]\} \quad \text{where} \quad \beta = \{x = 6\}(2*x) = 12 \]
      \[ = \{\{x = 6, y = 12\}\} \]

2. Let \( S = \text{if } x > 0 \text{ then } x := x+1 \text{ else } y := 2*x \text{ fi}. \)
   a. If \( \sigma(x) = 8 \), then \( \sigma(x > 0) = T \), so \( M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto 9]\} \)
   b. If \( \sigma(x) = 0 \), then \( \sigma(x > 0) = F \), so \( M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto \sigma(2*x)]\} = \{\sigma[y \mapsto 0]\} \)
   c. If \( \sigma(x) > 0 \) then \( M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\} \)
      If \( \sigma(x) \leq 0 \) then \( M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto 2 \times \sigma(x)]\} \)

3. Let \( S = \text{if } x > 0 \text{ then } x := x/z \text{ fi} = \text{if } x > 0 \text{ then } x := x/z \text{ else skip fi} \)
   a. If \( \sigma = \{x = 8, z = 3\} \), then \( \sigma(x > 0) = T \), so \( M(S, \sigma) = M(x := x/z, \sigma) = \{\sigma[x \mapsto \alpha]\} \) where \( \alpha = \sigma(x/z) \)
      \[ = \sigma[x \mapsto 8/3] = \sigma[x \mapsto 2] \], since integer division truncates.
   b. If \( \sigma = \{x = -2, z = 3\} \) then \( \sigma(x > 0) = F \), so \( M(S, \sigma) = M(\text{skip}, \sigma) = \{\sigma\} \).

4. Let \( W = \text{while } x < 3 \text{ do } S \text{ od} \) where \( S = x := x+1; y := y*x. \)
   a. For arbitrary \( \tau \),
      \[ M(S, \tau) = M(x := x+1; y := y*x, \tau) \]
      \[ = M(y := y*x, \tau[x \mapsto \tau(x)+1]) \]
      \[ = \{\tau[x \mapsto \tau(x)+1][y \mapsto a]\} \quad \text{where} \quad a = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1) \]
   b. If \( \sigma \models x = 4 \land y = 1 \), then \( \sigma(x < 3) = F \) so \( M(W, \sigma) = \{\sigma\} \).
   c. If \( \sigma \models x = 1 \land y = 1 \), then \( \sigma(x < 3) = T \) so we have at least one iteration to do. Let \( \sigma_0 = \sigma \),
      let \( \sigma_1 = M(S, \sigma_0) = \sigma_0(y) \times (\sigma_0(x)+1) \), and let \( \sigma_2 = M(S, \sigma_1) = \sigma_1(y) \times (\sigma_1(x)+1) \). Then
      \[ \sigma_0 = \sigma[x \mapsto 1][y \mapsto 1] \]
      \[ \sigma_1 = M(S, \sigma_0) = \sigma_0[x \mapsto \sigma_0(x)+1][y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2] \]
      \[ \sigma_2 = M(S, \sigma_1) = \sigma_1[x \mapsto 2+1][y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6] \]
      Since \( \sigma_0 \) and \( \sigma_1 \models x < 3 \) but \( \sigma_2 \models x \geq 3 \), we have \( M(W, \sigma) = \{\sigma_2\} = \{\sigma[x \mapsto 3][y \mapsto 6]\}. \)
Runtime Errors

5. \( M(S, \sigma) = M(x := y/b[x], \sigma) = \{ \sigma[x \mapsto y] \} \) where \( y = \sigma(y/b[x]) = 13/\sigma(b)(a) = \bot_e \)
   
   \( \text{iff } \sigma(b)(a) = \bot_e \) or \( \sigma(b)(a) = 0 \)
   
   \( \text{iff } (a \text{ is out of range for } \sigma(b)) \) or \( (\sigma(b)(a) = 0) \)

   \( \) (\( b[x] \) fails if \( x \) is out of range)

   \( \text{iff } (a < 0 \text{ or } a \geq 4) \) or \( (\sigma(b)(a) = 0) \)

   \( \) (\( \sigma(b) \) has size 4)

   \( \text{iff } (a < 0 \text{ or } a \geq 4) \) or \( (\alpha = 1) \)

   \( \) (\( b[1] \) is the only element = 0)

\( \text{iff } \neg(a = 0, 2, \text{ or } 3) \)

6. \( M(S, \sigma) = M(y := y/sqrt(b[x]), \sigma) = \{ \sigma[y \mapsto \beta] \} \) where \( \beta = (\sigma(y)/sqrt(y)) = (8/sqrt(y)) \) and \( y = \sigma(b)(a) = \sigma(b)(a) \).

   \( \) So \( \beta = \bot_e \) and thus \( M(S, \sigma) = \{ \sigma[y \mapsto \bot_e] \} = \{ \bot_e \} \)

   \( \text{iff } y = \bot_e \) or \( y < 0 \) or \( sqrt(y) = 0 \)

   \( \) (\( b[x] \) fails, \( b[x] < 0 \), or \( sqrt(b[x]) = 0 \))

   \( \text{iff } (a \text{ out of range for } \sigma(b)) \) or \( y < 0 \) or \( sqrt(y) = 0 \)

   \( \) (\( y = \bot_e \) iff \( b[x] \) has a bad index)

   \( \text{iff } (a < 0 \text{ or } a \geq 4) \) or \( y = \sigma(b)(a) < 0 \) or \( sqrt(y) = 0 \)

   \( \) (\( \sigma(b) \) is of size 4)

   \( \text{iff } (a < 0 \text{ or } a \geq 4) \) or \( (\alpha = 0) \) or \( sqrt(y) = 0 \)

   \( \) (only \( b[0] < 0 \))

   \( \text{iff } (a < 0 \text{ or } a \geq 4) \) or \( (\alpha = 0) \) or \( (\alpha = 3) \)

   \( \) (only \( sqrt(b[3]) = sqrt(0) = 0 \))

   \( \text{iff } (a \leq 0 \text{ or } \geq 3) \)

   \( \) (combining terms)