Satisfaction, Validity, and State Updates

CS 536: Science of Programming, Fall 2021

A. Why

• A predicate is satisfied or unsatisfied relative to a state.
• A predicate is valid if it is satisfied in all states.
• State updates occur when we introduce new variables or change the values of existing variables.

B. Outcomes

At the end of today, you should

• Know how to check a predicate for satisfaction in a state, how to check a predicate for validity, and know how to update a state.

C. Questions

1. Say u and v stand for variables (possibly the same variable) and α and β are values (possibly equal). When is \( \sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha] \)? Hint: There are four cases because maybe \( u \equiv v \) and maybe \( \alpha = \beta \).

2. Let \( \sigma(b) = (7, 5, 12, 16) \). Assume out-of-bound indexes cause runtime errors.
   a. Does \( \sigma \models \exists k . 0 \leq k \land k+1 < \text{size}(b) \land b[k] < b[k+1] \)? If so, what was your witness value for \( k \)?
   b. Does \( \sigma \models \exists k . 0 \leq k-1 \land k+1 < \text{size}(b) \land b[k-1] < b[k] < b[k+1] \)? If so, what was your witness value for \( k \)?
   c. Does \( \sigma \models \forall k . 0 \leq k < 4 \rightarrow b[k] > 0 \)?
   d. If \( \sigma(k) = -5 \), then does \( \sigma \models \exists k . b[k] > 0 \)?

3. For each of the situations below, fill in the blanks to describe when the situation holds.
   Fill in \( \_\_\_\_ 1 \) with “some”, “every”, or “this”
   Fill in \( \_\_\_\_ 2 \) with “some” or “every”
   Fill in \( \_\_\_\_ 3 \) with “\( \sigma(x) \) must be undefined”, “\( \sigma(x) \) must be defined and \( \sigma \models p \)”, or “nothing of \( \sigma(x) \)”
   Fill in \( \_\_\_\_ 4 \) with “\( \models p \)” or “\( \not\models p \)”
   a. \( \sigma \models (\exists x \in U. p) \) iff for \( \_\_\_\_ \) state \( \sigma \) and \( \_\_\_\_ \) \( \alpha \in U, \sigma[x \mapsto \alpha] \) \( \_\_\_\_ \)
   b. \( \sigma \models (\forall x \in U. p) \) iff for \( \_\_\_\_ \) state \( \sigma \) and \( \_\_\_\_ \) \( \alpha \in U, \sigma[x \mapsto \alpha] \) \( \_\_\_\_ \)
   c. \( \sigma \models (\exists x \in U. p) \) requires \( \_\_\_\_ \).
   d. \( \sigma \models (\forall x \in U. p) \) requires \( \_\_\_\_ \).
   e. \( \sigma \not\models (\exists x \in U. p) \) iff for \( \_\_\_\_ \) state \( \sigma \) for \( \_\_\_\_ \) \( \alpha \in U, \sigma[x \mapsto \alpha] \) \( \_\_\_\_ \)
f.  $\sigma \not\models (\forall x \in U. \ p)$ iff for _____, state $\sigma$ for _____ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____ 4

g.  $\not\models (\forall x \in U. \ p)$ iff for _____ state $\sigma$, we have $\sigma$ _____ ($\forall x \in U. \ p$).

h.  $\not\models (\exists x \in U. \ p)$ iff for _____ state $\sigma$, we have $\sigma$ _____ ($\exists x \in U. \ p$).

i.  $\not\models (\forall x \in U. \ p)$ iff for _____ state $\sigma$, and for _____ $\alpha \in U$, we have $\sigma[x \mapsto \alpha]$ _____ 4

j.  $\models (\exists x \in U. (\forall y \in V. \ p))$ iff for _____ state $\sigma$, for _____ $\alpha \in U$, and for _____ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ 4

k.  $\not\models (\exists x \in U. (\forall y \in V. \ p))$ iff for _____ state $\sigma$, for _____ $\alpha \in U$, and for _____ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta] [\models | \models ] p$.

l.  $\models (\forall x \in U. (\exists y \in V. \ p))$ iff for _____ state $\sigma$, for _____ $\alpha \in U$, and for _____ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta] [\models | \models ] p$.

m.  $\not\models (\forall x \in U. (\exists y \in V. \ p))$ iff for _____ state $\sigma$, for _____ $\alpha \in U$, and for _____ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ 4

4. Let $p = \exists y . \forall x . f(x) > y$, and let $q = \forall x . \exists y . f(x) > y$. (As usual, assume a domain of $\mathbb{Z}$.)

a.  Is it the case that (regardless of the definition of $f$), if $p$ is valid then so is $q$? If so, explain why. If not, give a definition of $f(x)$ and show $\models p$ but $\not\models q$.

b.  (The converse.) Is it the case that (regardless of the definition of $f$), if $q$ is valid then so is $p$? If so, explain why. If not, give a definition of $f(x)$ and show $\models q$ but $\not\models p$. 