A. Problems [60 points total]

Class 7: Sequential Nondeterminism

1. [13 = 3 + 3 + 7 points] Let DO be the nondeterministic loop
   
   \[
   \text{do } x \neq 0 \rightarrow x := x-1; \ y := y+1 \ \square \ x \neq 0 \rightarrow x := x-1; \ y := y+2 \ \text{od}
   \]

   a. Let’s figure out what a typical loop iteration does over an arbitrary state \( \sigma = \{x = \alpha, y = \beta\} \).

   Assume \( \alpha \geq 2 \) and calculate the two states we can be in after a single iteration of the loop.

   I.e., what are \( \sigma' \) and \( \sigma'' \) such that if \( \Sigma' = \{\sigma', \sigma''\} \), then \( \langle \text{DO}, \sigma \rangle \rightarrow^3 \langle \text{DO}, \tau \rangle \) where \( \tau \in \Sigma' \)?

   b. Repeat part (a) but for two iterations to get three possible final states.

   c. Generalize parts (a) and (b) to \( \kappa \) iterations where \( 1 < \kappa \leq \alpha \). I.e., what is \( \Sigma' \) such that \( \langle \text{DO}, \sigma \rangle \rightarrow^{3\kappa} \langle \text{DO}, \tau \rangle \) iff \( \tau \in \Sigma' \)?

Class 8: Hoare Triples, pt 1

The questions below have the form “If \( X \), then \( Y \) _____ occur”. To answer them, fill in the blank with “must”, “can’t”, or “may or may not”.

- **Must occur** means \( X \) implies \( Y \). (E.g., if \( x > 1 \), then \( x > 0 \) must occur.)

- **Can’t occur** means \( X \) implies \( \neg Y \). (E.g., if \( x > 1 \), then \( x < -3 \) can’t occur.)

- **May or may not occur** means that both \( X \land Y \) and \( X \land \neg Y \) can happen (one at a time, of course). (E.g., if \( x > 1 \), then \( y = 0 \) may or may not occur.)

You’re not required to justify your answer, though you can if you want to (and you should be able to if asked in to in an exam).

2. [30 = 15 * 2 points] Below, assume throughout that \( \sigma \neq \bot \) and that \( S \) might be deterministic or nondeterministic unless said otherwise.

   a. If \( \sigma \models \{p\} S \{q\} \) and \( \sigma \neq p \), then \( \bot \in M(S, \sigma) \) _____ occur.

   b. If \( \sigma \models \{p\} S \{q\} \) and \( \sigma \neq p \), then \( M(S, \sigma) - \{\bot\} \models q \) _____ occur.

   c. If \( \sigma \models \{p\} S \{q\} \) and \( \sigma \models p \), then \( \bot \in M(S, \sigma) \) _____ occur.

   d. If \( \sigma \models \{p\} S \{q\} \) and \( \sigma \models p \), then \( M(S, \sigma) - \{\bot\} \models q \) _____ occur.

   e. If \( \models_{\text{tot}} \{p\} S \{q\} \) then \( \models_{\text{tot}} \{p\} S \{T\} \) _____ occur.
f. If \( \models_{\text{tot}} \{ p \} S \{ T \} \) then \( \models_{\text{tot}} \{ p \} S \{ q \} \) _____ occur.

g. If \( \sigma \not\models \{ p \} S \{ q \} \) and \( S \) is deterministic, then \( \sigma = p, \bot \not\in M(S, \sigma) \), and \( M(S, \sigma) \models \neg q \) _____ occur.

h. If \( \bot \not\in M(S, \sigma) \), \( M(S, \sigma) \not\models q \), and \( S \) is deterministic, then \( M(S, \sigma) \models \neg q \) _____ occur.

i. If \( \bot \not\in M(S, \sigma) \), \( M(S, \sigma) \not\models q \), and \( S \) is nondeterministic, then \( M(S, \sigma) \models \neg q \) _____ occur.

j. If \( M(S, \sigma) \models \neg q \), \( \tau \in M(S, \sigma) \), and \( S \) is nondeterministic, then \( \tau \models q \) _____ occur.

k. If \( \sigma \models \{ p \} S \{ q \} \), then \( \sigma \models \{ p \} S \{ \neg q \} \) _____ occur.

l. If \( \sigma \not\models_{\text{tot}} \{ p \} S \{ q \} \) and \( S \) is deterministic, then \( \sigma \models \{ p \} S \{ \neg q \} \) _____ occur.

m. If \( \sigma \not\models_{\text{tot}} \{ p \} S \{ q \} \) and \( S \) is nondeterministic, then \( \sigma \models \{ p \} S \{ \neg q \} \) _____ occur.

n. If \( \sigma \not\models \{ p \} S \{ q \} \) and \( S \) is deterministic, then \( \sigma \models_{\text{tot}} \{ p \} S \{ \neg q \} \) _____ occur.

o. If \( \sigma \not\models \{ p \} S \{ q \} \) and \( S \) is non-deterministic, then \( \sigma \models_{\text{tot}} \{ p \} S \{ \neg q \} \) _____ occur.

Class 9: Hoare Triples, pt 2

3. [2 points] Study the triple \( \{ ??? \} x := b*b - 4*a*c \{ 0 \leq x \rightarrow ( \text{sqrt}(x) \text{ is defined}) \} \). Using backward assignment, what can we use for the precondition of the triple?

4. [3 points] Study the two triples \( \{ p \} x := n; y := m \{ p \land x = n \land y = m \} \) and \( \{ 1 \leq x*y \leq n*m \} S \{ q \} \). Find a predicate \( p \) that makes it possible to join the two triples into a sequence.

5. [4 = 2*2 points] Let \( p_0 \rightarrow p, p \rightarrow p_1, q_0 \rightarrow q, \) and \( q \rightarrow q_1 \) all be valid. From \( \{ p \} S \{ q \} \), there are four triples of the form \( \{ p_i \} S \{ q_j \} \) that get by replacing \( p \) by \( p_0 \) or \( p_1 \) and \( q \) by \( q_0 \) or \( q_1 \).

a. If \( \sigma \models \{ p \} S \{ q \} \), which of the four triples \( \sigma \models \{ p_i \} S \{ q_j \} \) is/are also satisfied by \( \sigma \) (under \( \models \))? Briefly justify.

b. If \( \sigma \models_{\text{tot}} \{ p \} S \{ q \} \), which of the four triples \( \sigma \models \{ p_i \} S \{ q_j \} \) is/are also satisfied by \( \sigma \) (under \( \models_{\text{tot}} \))? Briefly justify.

6. [8 = 4*2 points] Say \( \sigma \models \{ p_1 \} S \{ q_1 \} \) and \( \sigma \models \{ p_2 \} S \{ q_2 \} \).

a. Does \( \sigma \models \{ p_1 \land p_2 \} S \{ q_1 \land q_2 \} \)? Justify briefly.

b. Does \( \sigma \models \{ p_1 \lor p_2 \} S \{ q_1 \land q_2 \} \)? Justify briefly.

c. Does \( \sigma \models \{ p_1 \lor p_2 \} S \{ q_1 \lor q_2 \} \)? Justify briefly.

d. Does \( \sigma \models \{ p_1 \land p_2 \} S \{ q_1 \lor q_2 \} \)? Justify briefly.