Types, Expressions, States, Quantified Predicates
CS 536: Science of Programming, Fall 2021
Due Tue Sep 14, 11:59 pm

A. Formatting and Submitting Your Work

• Remember to use a word processor to write out your answers. Quantified variables range over $\mathbb{Z}$ unless otherwise specified.

B. Problems [60 points total]

Class 3: Types, Expressions, and Arrays

1. (6 = 3 * 2 points) For each of the following, is the expression legal or illegal according to the syntax we’re using. If illegal, why? If legal, what is the type of the resulting expression?
   a. $(x < y \ ? x : F)$ // Assume $<$ works on integers, not booleans
   b. $b[0] + b[1][1]$ // Assume $b$ is 2-dimensional
   c. $\text{match}(b1, b2, n)$ // Assume match asks if the first $n$ elements of $b1$ and $b2$ match
      // (Assume $b1$ and $b2$ are one-dimensional.)

2. (6 = 3 * 2 points) For each of the following are well-formed states? For the ones that aren’t, why?
   a. $\{x = (2), y = 4\}$
   b. $\{u = (3, 4), v = 0, w = u[1]\}$
   c. $\{r = \text{one}, s = \text{four}, t = r + s\}$

3. (4 = 2 * 2 points) Let $\sigma = \{x = 2, b = \beta\}$ where $\beta = (\text{five, two plus two, 6})$.
   a. Rewrite $\sigma$ giving the value of $b$ as a set of ordered pairs.
   b. Rewrite $\sigma$ giving the value of $b$ as separate bindings for $b[0], b[1],$ etc.

4. (6 = 3 * 2 points) Let $\varphi = x = y \cdot z \wedge y = 3 \cdot z \wedge z = b[0] + b[2] \wedge 3 < b[1] < b[2] < 6$. Complete the definition of $\sigma = \{x = ____, y = ____, z = 5, b = __________\}$ so that $\sigma \models \varphi$.

5. (8 = 4 * 2 points) Take the expression $0 \cdot b[b[j]]$. For each state below, is it well-formed and proper for the expression? And if so, does the expression terminate correctly (and with what result)? If not, why?
a. \{j = 0, b = (3, 2, 5, 4), c = (3), d = 8\}
b. \emptyset
c. \{j = 0, b = 0\}
d. \{b = (0), k = 0\}

Class 4: State Updates, Satisfaction of Quantified Predicates

6. (6 = 2 * 3 points) Let \(\sigma = \{x = 2, y = 4, b = (-1, 0, 4, 2)\}\).
   a. Is there a difference between \(\sigma[z \mapsto 1]\) and \(\sigma \cup \{(z, 1)\}\)? Justify your answer (very briefly).
   b. Repeat, on \(\sigma[x \mapsto 5]\) and \(\sigma \cup \{(x, 5)\}\)?

7. (9 = 3 * 3 points) Recall how satisfaction of quantified predicates and state updates are defined.
   a. Does \(\{x = 4, y = 6, b = (4,2,8)\} \models (\exists x. \exists j. b[j] < x < y)\)? If not, why?
   b. Does \(\{x = 0, y = 7, b = (4,2,8)\} \models (\forall x. \forall k. 0 < k < 3 \rightarrow x < b[k])\)? If not, why?
   c. Does \(\{x = 0, y = 7, b = (4,2,8)\} \models (\forall x. \forall k. 0 < k < 3 \land x < b[k])\)? If not, why?

8. (9 = 3 * 3 points) In English, explain briefly when each of the following holds.
   a. \(\not\models (\forall x \in U. (\exists y \in V. P(x, y)))\)
   b. \(\not\models (\forall y. (\exists x \in U. P(x, y)) \rightarrow (\exists y \in U. Q(x, y)))\)
   c. \(\not\models (\exists x \in U. (\forall y \in V. P(x, y)))\)

9. (6 points) Write a definition for a predicate function \(P(b, c, d, s, t) = \ldots\) such that every element in \(b[c], b[c+1], \ldots, b[d-1]\) equals some element in \(b[s], b[s+1], \ldots, b[t-1]\). E.g., in a state where \(b = (0, 3, 7, 2, 1, 4, 2, 2, 4)\), we have \(P(b, 6, 9, 3, 5)\) because 2, 2, and 4 appear in 2, 1, and 4. On the other hand, \(P(b, 3, 5, 6, 9)\) is false because 2, 1, and 4 don’t all appear in 2, 2, and 4.
   If any of \(c, d, s, t\) are illegal as indexes for \(b\), have \(P\) return false. Feel free to write helper predicate functions if it makes your life easier.