Logic Review

CS 536: Science of Programming, Fall 2023

New due date: Fri Jan 27, 11:59 pm

~~***Due Wed Jan 25, 11:59 pm***~~

ver. Mon 2023-01-23, 19:40; p.3; 01-26: p.2

A. Why?

- We use propositions and predicates to write program specifications.
- Propositions and predicates can be related or manipulated syntactically or semantically.

B. Objectives

At the end of this homework, you should be able to

- Perform various syntactic operations and checks on propositions and predicates.
- Describe the difference between syntactic and semantic equivalence.
- Form proofs of propositions using some standard proof rules.
- Design predicate functions for simple properties on values and arrays.

C. Problems [60 points total]

Quantified variables range over $\mathbb{Z}$ unless otherwise specified.

1. [8=4*2 points] Which of the following mean(s) $p \rightarrow q$ and which mean $q \rightarrow p$?
   a. p is sufficient for q
   b. p only if q
   c. p if q
   d. p is necessary for q

2. [4=2*2 points] Let e_1 and e_2 be expressions.
   a. In general, does $e_1 \neq e_2$ imply $e_1 \not\equiv e_2$? If yes, briefly justify (a sentence or two is fine); if no, give a counterexample (specific values for e_1 and e_2 that show that this implication does not always hold).
   b. In general, does $e_1 = e_2$ imply $e_1 \equiv e_2$,? Again give a brief justification or counterexample.

3. [6=3*2 points] For each pair below, characterize the state as well- or ill-formed; if well-formed, is it proper? If proper, does the given expression evaluate successfully or cause a runtime error (and if so, how?)
   a. $\{v=5,\ z=6\}$ and $v+0*w$
   b. $\{v=-4,\ w=6\}$ and $sqrt(v)*sqrt(w)$
   c. $\{y=18,\ z=2\}$ and $y*y/(z+4)$

4. [6 points] The goal is to show that $p \wedge \neg(q \vee r) \to q \vee r \to \neg p$ is a tautology by proving it is $\Leftrightarrow T$. To do this, complete the proof of equivalence below using (only) the propositional logic rules (from Class 2). Be sure to include the names of the rules. There's more than one correct answer [just give one of them].

   $p \wedge \neg(q \vee r) \to q \vee r \to \neg p$ [2023-01-26]
   [you fill in] Defn $\to$
   [you fill in] Defn $\to$
   [and so on]

5. [6 points] Simplify $\neg(\forall x.(\exists y.x \le y) \vee \forall z.x \ge z)$ to a predicate that has no uses of $\neg$. Present a proof of equivalence. Use DeMorgan's laws for quantified predicates: $\neg\forall u.r \Leftrightarrow \exists u.\neg r$ and $\neg\exists u.r \Leftrightarrow \forall u.\neg r$. [2023-01-12]. Also use rules like "$\neg(e_1 \le e_2) \Leftrightarrow e_1 > e_2$ by negation of comparison".

6. [4=2*2 points] What is the full parenthesization of
   a. $p \wedge \neg r \vee s \to \neg q \wedge r \to \neg p \leftrightarrow \neg s \to t$?
   b. $\forall m.0<m<n \wedge \exists j.0 \le j<m \to b[0] \le b[j] \le b[m]$ *

7. [4=2*2 points] Give the minimal parenthesization of each of the following by showing what remains after removing all redundant parentheses. Hint: To avoid getting confused about which parentheses match each other, try rewriting the given parentheses with subscripts: $(_1$ and $)_1$ versus $(_2$ and $)_2$ and so on.
   a. $((\neg(p \vee q) \wedge r) \to (((\neg q) \vee r) \to ((p \vee (\neg r)) \wedge (q \wedge s))))$
   b. $(\exists j.(((0 \le j) \wedge (j<m)) \wedge (\forall k.(((m \le k) \wedge (k<n)) \to (b[j]<b[k])))))$. (This predicate asks "Is there a value in $b[0..m-1]$ < every value in $b[m..n-1]$?)
   c. $(\forall x.((\exists y.(p \wedge q)) \to (\forall z.(p \to (q \wedge r)))))$

* Leave $(0 \le j<m)$ as is; don't expand it to $((0 \le j) \wedge (j<m))$. Don't forget to parenthesize $(b[0])$, e.g.

8. [10 points total] Say whether the given propositions or predicates are $\equiv$ or $\not\equiv$. Briefly justify your answer.
   a. [2 points] Is $\forall x.p \to \exists y.q \to r \equiv ((\forall x.p) \to (\exists y.q)) \to r$?
   b. [3 points] Is $\exists x.p \wedge \exists y.(q \to r) \vee \exists z.r \to s \equiv \exists x.p \wedge (\exists y.q \to r) \vee (\exists z.r \to s)$?
   c. [3 points] Is $(\forall x.p \vee \forall y.q) \vee (\forall z.r) \to s \equiv \forall x.p \vee (\forall y.q) \vee \forall z.r \to s$?
   d. [2 points] Is $p \wedge q \vee \neg r \to \neg p \to q \equiv ((p \wedge q) \vee ((\neg r \to ((\neg p) \to q))))$?

9. [6=3*2 points] Say whether each of the following is a tautology, contradiction, or contingency. If it's a contingency, show an instance when the proposition is true and show an instance where it's false.
   a. $((p \to q) \to r) \to (p \to (q \to r))$.
   b. $(p \to (q \to r)) \to ((p \to q) \to r)$
   c. $(\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}. f(x,y) > 0) \to (\exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}. f(x,y) > 0)$.

   [2023-01-12] ~~Use DeMorgan's laws for quantified predicates: ¬∀u.r ⇔ ∃u.¬r and ¬∃u.r ⇔ ∀u.¬r.~~

10. [6 points] Write the definition of a predicate function $GT(b, x, m, k)$ that yields true iff $x > b[m], \ldots\ b[k]$. E.g., in the state $\{b = (1, 3, -2, 8, 5)\}$, $GT(b, 4, 0, 2)$ is true [2023-01-23: 2, not 3]; $GT(b, 0, 1, 2)$ is false. You can assume without testing that the indexes $m, \ldots\ k$ are all in range. If $k < m$, the sequence $b[m], b[m+1], \ldots, b[k]$ is empty and $GT(b, x, m, k)$ is true. (It's straightforward to write GT so that this is not a special case.)

    Remember, this has to be a predicate function, not a program that calculates a boolean value. Hint: Check the discussion in the Class 2 notes about trying to translate programs to predicates.