Class 12  Tue Mar 2

Syntactic Substitution

$\text{wp}(x_i = e, Q(x)) = Q(e)$

Syntactic substitution takes $Q(x)$ to $Q(e)$

$\text{wp}(x_i = x+1, x \neq y) \quad y[x+1/x] = y$

$p \text{and} [\text{expr}/\text{var}] \quad \text{expr}[\text{expr}/\text{var}]$

$p[e/x], e_1[e/x] \quad p_0 \text{ if } p_e$

$p$ with $e$ for $x$ $e, w/e$ for $x$

$p[x\neq e]$

$p[x = e]$

Substitute $\text{expr} - \text{straight forward}$

$e, [e/x]$ scan $e$, look for $x$, replace by $(e)$

$x + (x*y + 1)$

$x + z$

$(x + z)[x+1/x] = (x+1)x + z$ (can drop parenz if redundant)

$(x+z) \[x\neq y/x] = x + y + z$

$\text{wp} x + 1 < z$

$(x + 5) [6/x] = 6 + 5$

$\text{wp} - 1$
e₁, [e/x] easy pred [e/x]
easy if pred has no ∀x ∃x
p[e/x] harder w/ them.
T[e/x] = T
y[e/x] = y if x ≠ y
(e₁ < e₂)[e/x] = e₁[e/x] < e₂[e/x]

(∀x p₁)[e/x] = p₁[e/x] ∨ p₂[e/x]

∃x (¬ p)[e/x] = ¬ (p[e/x])

Q p means ∀x or ∃x

(∀y p)[e/x] depends on x but also y, and on vars of e and p

Subst. only replaces non-local vars
bound = local

each occurrence of a variable in a pred. is either bound or in scope of a Q
or unbound = free

(∀x, f(x) ≥ g(x)) = bound, f unbound
Each occ. is either free or bound

\[ p = \forall x \forall y. \forall x. x + f(y) = 3 \]

\( \text{free} \quad \text{bound} \quad \text{free} \)

\[ p[e/x] \text{ only replace free occ's of } x. \]

\[ p[a + 2/x] = a + 2 \forall y. \forall x. x + f(y) = 3 \]

bound - not replaced.

Don't subst. for local occ's b/c those aren't the same variable - they just look similar.

\[ \text{if } (\exists x. x \geq 3) \text{ then } \exists z. z \geq 3 ? \]

\[ (\exists y. y \geq 3) \text{ then } \exists y. y \geq 3 \]

Shouldn't matter which local vars you use.

\[ p(x) = x^3 + bx \]

\[ p(y) = y^3 + by \]

same const. b
In a predicate,
Each individual occ. of $x$ is either free or bound

"$x$ is free in $p" \equiv \text{free occ. of } x \text{ in } p$
"$x$ is bound in $p" \equiv \exists \text{ bound occ. of } x \text{ in } p$

$x$ can be both free & bound in $p$

$x \not\in y \land \exists z, x \in y \quad x \text{ free or } x \text{ bound in } p$

$x \text{ free in } p \iff x \text{ bound in } p$

If $x$ occurs at all in $p$, then $x$ is free in $p$
or $x$ is bound in $p$

$x$ neither free nor bound in $p \iff x \not\in p$

Substitution cases

case 1 \text{ no quantifier in } p

\text{case 2 } (\forall x.p)[e/x] \equiv \forall x.p$

\text{same}

all occ. of $x$ are bound

so we don't replace then.
case 3 \((\forall y. p)[e/x] \neq y\) and \(y\) doesn't occur in.

\[\exists \, \forall y. (p[e/x])\]

\[\exists y. (y > x)[y+1/x] = \exists y. (y > x)[y+1/x]\]

\[\exists \exists y, y > 12\]

case 4 \((\forall y. p)[e/x] \neq y\) and \(y\) occurs in.

can't use \(\exists y. (p[e/x])\) because

\[\exists z, z > x\]

\[\exists y. (y > x)[y+1/x] \neq \exists y, y > y+1\]

renaming loc. var.

trying to some thing like

\(\exists z, z > x\) \(y+1/x\) = \exists z, z > y+1

replace bound var. \(y\) with fresh = unused var.
case 4

\[(\forall y.p)[e/x] \times \forall y, y \text{ occuring}\]

\[\text{find fresh var } \exists y, \exists x, \text{ not in or e already}\]

\[(\exists z, p[z/y])[e/x] \text{ case 3} \]

\[\exists y. y > x + 5 \prod 2x - y/x\]

\[\exists z. (y > x + 5) \prod 2y/x \]

\[\exists z. z > (2y - y) + 5 \]

\[\exists z. z > (2y - y) + 5 \text{ replaces } x \text{ with } y\]

\[\text{came from } \prod 2x - y/x\]

Any occ of renamed

Var should be

free in result of \((\forall y.p)[e/x]\)
Examples practice \leftarrow \rightarrow

\begin{align*}
(x + i*b + c = 0) & \Rightarrow (b+c/c) \\
(x + (i+t)*b + c = 0) & \Rightarrow (b+c/c) \\
x + (i+t)*b + (b+c) & = 0
\end{align*}

\begin{align*}
p \equiv \exists x, x cy \& x^2 \geq y + tk & \text{is bound in } p \\
p[5/x] & \equiv \exists x, x cy \& x^2 \geq y + tk \equiv p \\
p[5/y] & \equiv \exists x, x cy \& x^2 \geq 5 + tk \\
p[5/z] & \equiv p \ (z \mathrm{~not~free~in~} p) \\
& \text{not in practice} \quad (z \text{~doesnt occur at all}) \\
p[x^2/y] & \equiv \exists x, x cy \& x^2 \geq y + tk \ [x^2/y] \\
& \equiv \exists x, x cy \& x^2 \geq y + tk \ [x^2/y] \\
& \equiv x cy \& x^2 \geq y + tk \ [x^2/y]
\end{align*}