Lambda (λ) calculus

Very simple + powerful PL
"Universal" model of functional PLs
Simple to reason about!

$$M ::= x \mid \lambda x. M \mid M\, M$$

That's all!

function "abstraction" — application

Examples

$\lambda x. x$	Identity
$\lambda x. \lambda y. x$	First
$\lambda x. \lambda y. y$	Second
$\lambda f. \lambda x. f\, x$	Apply
$\lambda f. f\, f$	Self-apply — no types, so this is fine!
$\lambda g. \lambda f. \lambda x. g(f x)$	Composition

Abstractions are right-associative, app. left-assoc.

$$\lambda x. \lambda y. \lambda z. x\, y\, z \equiv \lambda x. (\lambda y. (\lambda z. (x y) z))$$

Application has higher precedence than abstraction

$$\lambda x. x y\ \lambda z. x z \equiv \lambda x. ((x\, y)(\lambda z. (x z)))$$

Free + bound variables

Abstractions bind variables
- A var. is bound if it's in the scope of a binding in AST:

$$\lambda x. \;\ldots\; x$$

$$\lambda z. (\lambda x. x z)(\lambda y. y z) \qquad (\lambda x. x \underset{F}{y})(\lambda y. \underset{F}{x} y) \qquad \lambda x. \underset{F}{y}(\lambda y. y \underset{F}{z}\ (\lambda z. x z))$$

$$\lambda x. x (\lambda x. x)$$

Substitution

$[N/x]M$ Substitute N for free x's in M

$[y/x]\, x = y$

$[y/x]\, (x\ z\ (\lambda x.x)) = y\ z\ (\lambda x.x)$ Not free!

$[y/x]\, (\lambda z.\, xx) = \lambda z.\, yy$

$[y/x]\, (\lambda y.\, yx) \neq \lambda y.\, yy$ y is "captured"

This refers to some outer y

α-equivalence

"Names of bound variables don't matter"

$\lambda x.\, x \equiv_\alpha \lambda y.y \equiv_\alpha \lambda z.z$

$\lambda x\, \lambda y.\, xy \equiv_\alpha \lambda a.\lambda b.\, ab$

Examples

$\lambda x.\, \lambda y.\, yyx \overset{?}{\equiv} \lambda g.\, \lambda f.\, ggf$ No

$\lambda x.\, x\ (\lambda y.z\ y) \overset{?}{\equiv} \lambda i.\, i\ (\lambda j.\, k\ j)$ No

$\lambda x.\, \lambda y.\lambda z.\, xzy \overset{?}{\equiv} \lambda a.\, \lambda y.\, \lambda b.\, a\, b\, y$ Yes

Preventing capture: If we're substituting a term w/a free var y into $\lambda y.M$, α-convert (convert to α-equivalent term)

$[y/x]\, (\lambda y.\, yx) = [y/x]\, (\lambda z.\, zx) = \lambda z.\, zy$

$[N/x]\ x = N$

$[N/x]\ y = y \quad x \neq y$

$[N/x]\, (\lambda x.M) = \lambda x.\, M$

$[N/x]\, (\lambda y.\, M) = \lambda y.[N/x]M \quad y \notin FV(N)$ free variables

$[N/x]\, (M_1\, M_2) = [N/x]M_1\ [N/x]M_2$ (If $y \notin FV(N)$, α-convert so it isn't)

Computing in λ-calculus

β-reduction

$$(\lambda x. M)\, N \to_\beta [N/x]M$$

$$
\begin{aligned}
& (\lambda x. \lambda y. x\, y)\,(\lambda x. x\, y)\,(\lambda y. y) \\
\to_\alpha\ & (\lambda x. \lambda z. x\, z)\,(\lambda x. x\, y)\,(\lambda y. y) \\
\to_\beta\ & (\lambda z. (\lambda x. x\, y)\, z)\,(\lambda y. y) \\
\to_\beta\ & (\lambda x. x\, y)\,(\lambda y. y) \\
\to_\beta\ & (\lambda y. y)\, y \\
\to_\beta\ & y
\end{aligned}
$$

η (Eta) reduction

If $x \notin FV(M)$, then $\lambda x. M\, x \to_\eta M$

Sort of like `let add1 l = List.map ((+) 1) l`

$\equiv$ `let add1 = List.map ((+) 1)`

$(\lambda y. \lambda x. y\, x)\, w\, z \to_\beta (\lambda x. w\, x)\, z \to_\beta w\, z$) Can reduce in many ways/orders

$\quad \to_\eta (\lambda y. y)\, w\, z \to_\beta w\, z$

Normal form - No β-reductions are possible

Not every expression "has" (can be reduced to) a normal form!

$$(\lambda x. x\, x)\,(\lambda x. x\, x) \to_\beta (\lambda x. x\, x)\,(\lambda x. x\, x) \to_\beta \dots$$

$(\lambda x. x)\,((\lambda y. y)\, z) \to_\beta (\lambda y. y)\, z \to_\beta z$ ← Call by name (lazy)

$(\lambda x. x)\,((\lambda y. y)\, z) \to_\beta (\lambda x. x)\, z \to_\beta z$ ← Call by value (eager)

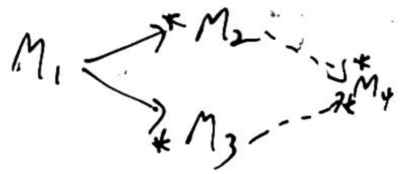
CBV may not terminate even if M has a normal form

$$(\lambda x.z)\ ((\lambda x.xx)(\lambda x.x\ x)) \xrightarrow[\beta]{CBN} z$$

However! If 2 reductions terminate, both are guaranteed to have the same result (i.e, normal forms are unique)
"Church-Rosser Property/Theorem"

0 or more reductions

Diamond Property: If $M_1 \xrightarrow[\beta]{}^* M_2$ and $M_1 \xrightarrow[\beta]{}^* M_3$, then $\exists$ some M_4 such that $M_2 \xrightarrow[\beta]{}^* M_4$ and $M_3 \xrightarrow[\beta]{}^* M_4$



Proof outline:

1 step

Lemma: If $M_1 \xrightarrow[\beta]{} M_2$ and $M_1 \xrightarrow[\beta]{} M_3$, then there exists M_4 such that $M_2 \xrightarrow[\beta]{}^* M_4$ and $M_3 \xrightarrow[\beta]{}^* M_4$

Corollary: If $M_1 \xrightarrow[\beta]{}^* M_2$ and $M_1 \xrightarrow[\beta]{}^* M_3$ and M_2 and M_3 are in normal form, then $M_2 = M_3$