CS 330 - Class 13, Thu Mar 4

Induction & Recursion

Exam 1 on Tue Mar 9

- Study guide on website

Rewrote those notes after class - incorporated some things, cleaned up/rephrased other things.

Question about Homework

If $f \circ g$ is onto then $f$ must be onto.

- Proof by contradiction: assume $f$ isn't onto
- So there are $x_1$ and $x_2$ where $x_1 \neq x_2$ and $f(x_1) = f(x_2)$
- But then $g(f(x_1)) = g(f(x_2))$, so $(f \circ g)(x_1) = (f \circ g)(x_2)$.
- Therefore $f \circ g$ is not onto
- But this contradicts the premise that $f \circ g$ is onto.
- Since assuming $f$ is not onto leads to a contradiction, we know $f$ is (not not) onto.
**Induction & Recursion**

- Mathematical induction is a technique for proving conjectures with form $\forall x \in \mathbb{N} \ P(x)$.

**Rule of inference for mathematical induction**

\[
\begin{align*}
P(1) & \quad \text{[base step]} \\
\forall m \ (P(m) \rightarrow P(m+1)) & \quad \text{[inductive step]} \\
\hline \\
\forall n \ P(n) & \quad \text{[Conclusion]} \\
\end{align*}
\]

*Base doesn’t have to be 1 — depends on context of particular problem*

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**How can we prove $P(2)$?**

- Start with proof of $P(1)$
- Instantiate $\forall m \ (P(m) \rightarrow P(m+1))$ to get a proof of $P(1) \rightarrow P(2)$.
- Apply modus ponens.

**How about proving $P(3)$?**

- Take the proof of $P(2)$.
- Instantiate the induction step, get proof of $P(2) \rightarrow P(3)$.
- Apply modus ponens.
How do we prove \( P(1000) \)?

- It's similar: We combine proofs of \( P(1) \), \( P(2) \rightarrow P(3) \), ..., \( P(999) \rightarrow P(1000) \).
- Ladder analogy

How do we prove \( P(x) \) for arbitrary \( x > 1 \)?

- Combine proofs of \( P(1) \), \( P(2) \rightarrow P(3) \), ..., \( P(x-1) \rightarrow P(x) \).
- Writing the induction step as \( \forall m \ (P(m) \rightarrow P(m+1)) \) lets us combine all of the (non-base) cases with one pattern.

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**Example:** Prove \( \forall n \ (2^0 + 2^1 + ... + 2^n = 2^{n+1} - 1) \) by induction

**Basis** \( P(0) \)

\[ 2^0 = 2^{0+1} - 1 \quad 1 = 2 - 1 \]

**Inductive step:** Show that \( P(n) \rightarrow P(n+1) \)

- Assume \( P(n) \): \( (2^0 + 2^1 + ... + 2^n) = 2^{n+1} - 1 \)
- So \( P(n+1) \) is \( (2^0 + 2^1 + ... + 2^{n+1}) = 2^{n+1+1} - 1 \) (i.e. \( 2^{n+2} - 1 \))
- If we add \( 2^{n+1} \) to the left side of the \( P(n) \) equation, we get \( (2^0 + 2^1 + ... + 2^n) + 2^{n+1} \), which matches \( P(n+1) \).
- To preserve equality, we should add \( 2^{n+1} \) to the right side of the \( P(n) \) equation and get \( (2^{n+1} - 1) + 2^{n+1} = 2 \times 2^{n+1} - 1 = 2^{n+2} - 1 \), which matches \( P(n+1) \) again.
• To review: We took \( P(n) \) and did some manipulation to infer (first the left side and then the right side of) \( P(n+1) \). So \( P(n) \rightarrow P(n+1) \), which is what we need for the inductive case of our goal.

• You can also see this result by using bitstrings
  • \((2^0 + 2^1 + \ldots + 2^n)_{10} = 111\ldots1_2 \) (with \( n+1 \) one bits)
  • \( 2^{n+1} = 1000\ldots0_2 \) (with \( n+1 \) zero bits)
  • Subtracting 1 from 1000\ldots0_2 gives us 111\ldots1_2.

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**Example:** Prove that \( \forall n \geq 5, 2^n > n^2 \)

• Base case is \( P(5) \) this time: \( 2^5 = 32 > 25 = 5^2 \)
• Inductive step is \( 2^n > n^2 \rightarrow 2^{n+1} > (n+1)^2 \)
• Assume \( 2^n > n^2 \). Left side of \( P(n+1) \) equation is \( 2^{n+1} \), which suggests we multiply \( 2^n \) by 2 (or add \( 2^n \))
• Multiplying both sides of \( P(n) \) equation gives us \( 2 \times 2^n > 2 \times n^2 \).
  • The left side is \( 2 \times 2^n = 2^{n+1} \), which is what we want.
  • The right side is \( 2n^2 \); we need it to be \( > (n+1)^2 \)
  • So we need \( 2n^2 > (n+1)^2 = (n^2 + 2n + 1) \).
  • Rearranging, we need \( n^2 - 2n - 1 > 0 \)

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• This holds if \( n \geq 3 \), so since \( n \geq 5 \), we're ok.
• Reviewing, \( 2^n > n^2 \) does imply \( 2^{n+1} > (n+1)^2 \), which is what we need for the inductive case.

\[ \text{Example: Fibonacci sequence} \]

• \( F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2} \) for \( n > 2 \)
• \( 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \)
• Prove \( P(k) \): \( F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k} \) for \( k \geq 1 \)
• Basis \( P(1) \): \( 2k-1 = 1 \) and \( 2k = 2 \), so we need \( F_1 = F_2 \), which is true (they're both 1).
• Inductive step: Show \( P(k) \rightarrow P(k+1) \).
  • Assume \( P(k) \) holds, so \( F_1 + F_3 + \ldots + F_{2k-1} = F_{2k} \).
  • We need \( F_1 + F_3 + \ldots + F_{2(k+1)-1} = F_{2(k+1)} \)
  • If we subtract the \( P(k) \) equation from the \( P(k+1) \) equation, we get
• On the left,

\[(F_1 + F_3 + \ldots + F_{2k-1} + F_{2(k+1)-1}) - (F_1 + F_3 + \ldots + F_{2k-1})\]

\[= F_{2(k+1)-1} = F_{2k+1}\]

• On the right,

\[F_{2(k+1)} - F_{2k} = (F_{2k+1} + F_{2k+1}) - F_{2k} = F_{2k+1}\]

• Since the two sides are equal, we find \(P(k+1)\) does hold.

• Question: What is the maximum number of pieces \(P(n)\) into which we can divide a circular pizza using \(n\) straight cuts? Let's run some experiments first.

1 cut, 2 pieces

2 cuts, 4 pieces

3 cuts, 7 pieces

4 cuts, 11 pieces
When we make a new cut, it adds a new area for each cut line it crosses, plus there's one new area after the last crossing.

- The *n*th cut can add at most \((n-1) + 1 = n\) new areas because it can cross at most \(n-1\) previous cut lines.
- Initially, \(P(0) = 1\), since we start with the whole pie.
- After \(n\) cuts, we have \(P(n) = n + P(n-1)\) areas
  So \(P(n) = (n + (n-1) + (n-2) + \ldots + 1) + 1\) extra for \(P(0)\).
- So \(P(n) = n(n+1)/2 + 1 = (n^2 + n)/2 + 1 = (n^2 + n + 2)/2\)
- Basis: 0 cuts: \(P(0) = (0^2 + 0 + 2)/2 = 1\ ✓
- Inductive step: Does \(P(n+1) = ((n+1)^2 + (n+1) + 2)/2\)?
  - \(P(n+1) = n+1 + P(n)\) 
    \[\text{[earlier observation]}\]
  - \(= n+1 + (n^2 + n + 2)/2\) 
    \[\text{[inductive hypothesis]}\]
  - \(= (2n+2 + n^2 + n + 2)/2\) \[\text{[combine into 1 fraction]}\]
  - \(= (n^2 + 3n + 4)/2\) \[\text{[collect terms]}\]
  - \(= (n^2 + 2n + 1 + n + 3)/2\) \[\text{[separate terms]}\]
  - \(= ((n+1)^2 + (n+1) + 2)/2\) \[\text{[which is what we wanted]}\]
All Horses are Brown!

- (Assuming I see a brown horse outside my window)
- Let \( P(n) = "\text{Given } n \text{ horses, they must be all brown}" \)
- Prove \( \forall n \in \mathbb{N} \ P(n) \) by induction.
- Base step: \( P(1) \) because I can see one brown horse.
- Inductive step: Show \( P(n) \rightarrow P(n+1) \)
  - Assume any \( n \) horses are always brown.
  - Given \( n+1 \) horses, horses numbered 1, 2, ..., \( n \) must be brown by inductive hypothesis. Similarly, horses numbered 2, 3, ..., \( n, n+1 \) must also be brown.
  - So then all \( n+1 \) horses are brown.

Where's the error?