CS 330 - Class 9, Tue Feb 18
Sets, Functions, Relations

From last time:
Set operations & relations $\cap \cup \setminus$, complement, $\subseteq$, $\subset$
Venn diagrams, Power sets
Ordered pairs, tuples, Cartesian products

Relations and Functions
Relations first (functions are special relations)

A binary relation $R$ of set $A$ to set $B$ is a subset of $A \times B$.
$a \in A$ and $b \in B$ are related by $R$ means $(a, b) \in R$
often write $a R b$ (negation is $a \not R b$)
($\emptyset$ and $A \times B$ are very boring [trivial] relations)

Example: Let $A = \{2, 4, 6\}$, $B = \{p, q\}$
\[
\{ (2, p), (4, q), (4, p) \}
\]
A relation **on a set** $A$ is a relation from $A$ to $A$.

Example: Let $A = \{1, 2, 3, 4\}$

$$\{(1,1), (1,2), (1,3), (1,4),
(2,4), (3,4), (4,1)\}$$

Note: $<, \leq, =, \neq$ are all examples of relations on numbers

A relation $R$ on a set $A$ is

- **Reflexive** if $a R a$
- **Symmetric** if $a R b$ implies $b R a$
- **Transitive** if $a R b$ and $b R c$ implies $a R c$
- **Irreflexive** if $a \not R a$
- **Antisymmetric** if $a R b$ and $b R a$ implies $a = b$

What properties do $<, \leq, =, \neq$ have?
**Representing a relation with a matrix** of 0's & 1's

Say $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_m\}$

$R$ is a relation from $A$ to $B$

Matrix $M$ represents $R$

$m_{ij} = 1$ if $(a_i, b_j) \in R$

$m_{ij} = 0$ if $(a_i, b_j) \not\in R$

Example: $A = \{a, b, c\}$, $B = \{3, 4\}$, $R = \{(a, 3), (b, 4), (c, 3), (c, 4)\}$

**Representing a relation with a directed graph**

Say $A = \{a_1, a_2, ..., a_n\}$

$R$ is a relation on $A$

Digraph $D$ represents $R$ means $D$ has an edge $a \rightarrow b$ iff $a \, R \, b$. (Note can have $a = b$)

Example: $A = \{a, b, c, d\}$, $R = \{(a, b), (b, a), (b, c), (c, c)\}$
Digraph representation of reflexive, symmetric, transitive, and antisymmetric relations?

reflexive

symmetric

transitive

antisymmetric

Matrix representation of reflexive, symmetric, transitive, and antisymmetric relations?

reflexive

symmetric

transitive

(check $M^2$)

antisymmetric
**Equivalence Relation**

The $=$ relation is reflexive, symmetric, and transitive (and antisymmetric)

An relation on $\mathbb{R}$ is an **equivalence relation** iff it's reflexive, symmetric, and transitive (weaker than $=$ without antisymmetry)

Two elements $a, b$ are **equivalent** if they're related

$a \sim b$ and $a = b$ are a couple of notations

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**Example:**

Equivalence on $\mathbb{N}$ if both even

\[ \{ (a, b) \in \mathbb{N}^2 \mid a - b \text{ divisible by } 2 \} \]

Generalizes to **congruence modulo $n$**

Written $a \equiv b \pmod{n}$

Holds iff $a - b$ divisible by $n$

\[ \{ (a, b) \in \mathbb{N}^2 \mid \exists k \in \mathbb{Z} \, (a - b = k \cdot n) \} \]

\[ 2 - 0 \equiv 1.2 \quad \text{even/odd is } \equiv \text{ mod } 2 \]

\[ 4 - 2 \equiv 1.2 \]

\[ 5 - 1 \equiv 2.2 \]

\[ 1 \equiv 4 \pmod{5} \]

\[ 2 \equiv 5 \]

\[ 3 \equiv 0 \]
Check: congruence mod $n$ is an equivalence relation

reflexive

\[ a \equiv a \pmod{n} \quad a - a = kn \text{ some } k \]

symmetric

\[ a \equiv b \pmod{n} \text{ so } a - b = kn \text{ some } k \]

\[ b - a = -kn \text{ so } b \equiv a \pmod{n} \]

transitive

\[ a \equiv b \pmod{n} \]

\[ a - b = k_1 n \text{ some } k_1, k_2 \]

\[ b \equiv c \pmod{n} \]

\[ b - c = k_2 n \quad a - c = (k_1 + k_2)n \]

\[ a \equiv c \pmod{n} \]

If $R$ is an equivalence relation on set $A$, and $x \in A$, then

the **equivalence class** $[x]_R$ of $x$ is \{ $y \in A \mid (x, y) \in R$ \}

Examples: If $R$ is equivalence mod 2 on $\mathbb{Z}$, then

$[0]_R = \{ ..., -6, -4, -2, 0, 2, 4, 6, ... \} \quad \text{all } x \in \mathbb{Z} \text{ where } x \text{ is even}$

$[1]_R = \{ ..., -5, -3, -1, 1, 3, 5, ... \} \quad \text{all } x \in \mathbb{Z} \text{ where } x \text{ is odd}$

If $R$ is equivalence mod 3 on $\mathbb{Z}$, then

$[0]_R = \{ ..., -9, -6, -3, 0, 3, 6, 9, ... \} \quad \text{remainder 0}$

$[1]_R = \{ ..., -8, -5, -2, 1, 4, 7, 10 \} \quad \text{remainder 1}$

$[2]_R = \{ ..., -7, -4, -1, 2, 5, 8, ... \} \quad \text{remainder 2}$

Every integer is in exactly one of these 3 classes!

\[ \text{mod } 3 \]
More generally, given an equivalence relation on \( A \), the set of equivalence classes partitions \( A \).

For a partition,

The union of the members = the original set

\[ 0_R \cup [1]_R \cup [2]_R = \mathbb{Z} \quad \bigcup_{a \in A} [a]_R = A \]

Two members are either identical or disjoint (no partial overlaps)

\[ [a]_R = [b]_R \iff (a, b) \in R \]

\[ [a]_R \cap [b]_R = \emptyset \iff (a, b) \notin R \]

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**Functions**

A function \( f \) from set \( A \) to set \( B \) is a relation from \( A \) to \( B \) where for each \( a \in A \), \( a \) is related via \( f \) to exactly one element of \( B \).

We write \( f : A \rightarrow B \) and \( f(a) = b \)

\( f \subseteq A \times B \) and \( \forall a \in A \) (\( \exists ! b \) (\( (a, b) \in f \))

\( A \) is the **domain** of \( f \); \( B \) is the **codomain** of \( f \).

The **range** of \( f \) is the set of \( B \) values assigned by \( f \).

(\( \text{So range} \subseteq \text{codomain} \))

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**Question:** If \( f : A \rightarrow B \) and \( B' \supseteq B \), then is \( f : A \rightarrow B' \) too?

\( \text{codomain} \)

\( \text{codomain} \)
Types of functions

**One-to-one (injection)**
No element of codomain is assigned to > 1 element of A: \( a \neq b \) implies \( f(a) \neq f(b) \)

**Onto (surjection)**
All elements of codomain are assigned by (range of \( f = \) codomain of \( f \))

**One-to-one correspondence (bijection)**
Both one-to-one and onto

**Partial:**
With \( f : A \to B \), for some \( a \in A \), \( f(a) \) is undefined

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**E.g. functions from \( A \to B \)**

- **bijection**
- injection & not surjection
- surjection & not injection
- neither injection nor surjection
- partial function
- not a function

*notes:
- target of > 1 arrow
- unused B value
- multiple arrows
- not all covered
- arrow for one B val.
- unused A value
- 1-1 not all covered
For a relation \( R \) in general, you can define an **inverse** \( R^{-1} \).

If \( R \) is from \( A \) to \( B \), then \( R^{-1} \) is from \( B \) to \( A \) where \( b \in R^{-1} a \) iff \( a \in R b \).

\[ R \quad \begin{array}{c|c}
    & b \\
    \hline
    a & R^{-1}
\end{array} \]

For a bijection \( f : A \to B \), the inverse \( f^{-1} \) is a function.

Domain of \( f^{-1} \) is range of \( f \) = codomain of \( f = B \).

Given \( b \), we have a \( f^{-1} b \) for exactly one \( a \), so \( b \in f^{-1} a \).

1-1 ness guarantees function-like behavior.

Onto guarantees total function vs partial function.

\[ \text{for exactly one } a \]

The **composition** of \( f : A \to B \) and \( g : B \to C \) is written \( g \circ f \) and is a function \( A \to C \) where \( (g \circ f)(a) = g(f(a)) \).

\[ \text{E.g. } f(x) = x^2, \ g(y) = 2 \ y, \ (g \circ f)(x) = ?? \]

\[ g(f(x)) = g(x^2) = 2 \ x^2 \]
**n-ary relations**

A relation on 3 sets A, B, C is a subset of $A \times B \times C$

Generalize to $A_1, A_2, ..., A_n$ (a relation of **degree** n)

A **relational database** is a relation of degree n

Has records with n fields (= an n-tuple)

```
record1 (a1, b1, c1)
record2 (a2, b2, c2)
record3 (a3, b3, c3)
```

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>b1</td>
<td>c1</td>
</tr>
<tr>
<td>a2</td>
<td>b2</td>
<td>c2</td>
</tr>
<tr>
<td>a3</td>
<td>b3</td>
<td>c3</td>
</tr>
</tbody>
</table>

**A primary key** is a field that uniquely specifies a tuple

E.g., $(s, n, r) \in \text{String} \times \text{N} \times \text{R}$, where the string determines the $\text{N} \times \text{R}$. ("Alex", 3, 15.8)

Can have > 1 tuple with 3 and 15.8 but only 1 tuple with "Alex"

**A composite key** is a collection of fields that uniquely specifies a tuple.

E.g., ("Logan", "argle", 2.5) and ("Logan", "bargle", 2.1)

both $\in \text{NameString} \times \text{IdString} \times \text{R}$

name and id string together determine the $\text{R}$ value
**Selection** w/ property P
Find all tuples for which P holds
E.g. select tuples where name begins with L

**Projection** w/ indexes $c \subseteq \{1, \ldots, n\}$
Take all tuples and toss out fields not in index set
E.g., project fields 1 and 2 from a 3-tuple

**Join** two databases R and S
Combines tuples from R and S based on common attributes