Theory Qualifier Exam

FAll 2016, CS Department, IIT

1. Given a list of students and their marks in a test find the first student such that the set of students who are alphabetically before her/him, have marks summing to $\geq W/2$, where W is the sum total of all marks.

   Can you do so in linear time? [Hint: Use median finding algorithm].

2. You have T time to complete n jobs , $j_1, j_2 \ldots j_n$. Task T_i gives a benefit of B_i and requires T_i units of time. Further you have 2 machines to schedule these tasks. Design and analyze an algorithm to determine the set of tasks that are to be scheduled and that optimize the total benefit. The total benefit B from choosing a set of tasks $\mathcal{J}$ is defined to be $\Sigma_{j_i \in \mathcal{J}} B_i$. Note that you cannot split a task or partially complete it. Your algorithm must be polynomial in n and T,

3. A *k-coloring* of an undirected graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. Consider the following "greedy" heuristic for graph coloring: Starting with an arbitrary vertex, color it with color 1; then, considering the other vertices in turn, color each vertex with color 1 whenever possible. When all such vertices are colored, pick an arbitrary uncolored vertex and color it with color 2; then, considering the other vertices in turn, color each vertex with color 2 whenever possible. Continue in this manner with colors 3, 4, and so on until all vertices are colored.

   (a) Briefly explain how to implement this heuristic in $O(|V| + |E|)$ time and space.

   (b) Prove that for every graph G there is an order of processing the vertices such that this heuristic yields an optimal coloring (that is, a coloring with as few colors as possible).

   (c) Construct a 2-colorable graph of 2^k vertices for which this heuristic, if it examines the vertices in an unfortunate order, will produce a coloring that uses $k + 1$ colors. (*Hints*: Consider the recursive construction of an acyclic graph of $2^k = 1 + (2^0 + 2^1 + \cdots + 2^{k-1})$ vertices; show that the resulting graph is 2-colorable, but that the heuristic may use $k + 1$ colors.)

4. Use the directed traveling salesperson tour problem to show that the problem of determine a shortest path without a cycle in a directed graph is NP-Hard. You can allow for the shortest path from a vertex v to itself be a cycle with only v repeating. The weights on the edges could be positive, zero or negative. [Hint: Use the transformation of weights $w(e) = C(e) - K$ where $C(e)$ is the cost of an edge in the TSP problem and K is a large constant].