

Written Theory Qualifier Exam

Your number: _____

Time limit: 2.5 hours. Use only the notes supplied by the Department

If you chose CS 530, solve problems 1, 2, 3, 4

If you chose CS 535, solve problems 1, 2, 5

SPRING 2012, CS DEPARTMENT, IIT

For every question, please write your answer in a clean and concise way. Use additional pages, start a new page with each problem and write only on one side of the paper.

Use procedures if you want – marking clearly what the parameters are and what they do, and with what running time in terms of its parameters. Unless the procedures are from the textbooks, write pseudocode for the procedures.

1. Suppose we are given two sorted arrays S and T , each with n items. Describe an $O(\log n)$ -time algorithm for finding the k th smallest key in the union of the keys from S and T (assuming no duplicates). Present pseudocode, argue correctness and analyze the running time.
2. Let $G = (V, E, c)$ be a weighted undirected graph where all the costs c_e , for $e \in E$, are strictly positive and distinct. Let T be a minimum spanning tree in $G = (V, E, c)$. Now suppose we replace the cost of each edge $e \in E$ by $c'_e = \sqrt{c_e}$, creating the instance $G' = (V, E, c')$. Prove or disprove: T is a minimum spanning tree in G' .

Now assume $s, t \in V$ are also given, and P is a shortest $s - t$ path in G . Prove or disprove: P is the shortest $s - t$ path in G' .

3. Classify the following language appropriately from amongst the following categories: (Regular languages, Context Free languages, Turing-recognizable languages). Give proofs:

(a) $\{0^n 1^m 0^p 1^r \mid n, m, p, r \geq 0 \wedge n = 2p \wedge r = 2m\}$.

4. Please use the numbering of theorems from the handout when using a particular theorem, and do not use without proving results not in the handouts.

Show that for any language A , if A is Turing-recognizable, and $A \leq_m \bar{A}$, then A is decidable.

5. Chapter 13 of CLRS (pages 308–338 in the 3rd edition) gives four basic operations on red-black trees that cause structural changes: node insertions, node deletions, rotations, and re-colorings. RB-INSERT and RB-DELETE use $O(1)$ rotations, node insertions, and node deletions to maintain the red-black properties, but they may make many more re-colorings. This problem is to prove

Theorem. *Any sequence of m RB-INSERT operations on an initially empty red-black tree causes $O(m)$ structural modifications in the worst case.*

- (a) Describe how to construct a red-black tree on n nodes such that RB-INSERT causes $\Omega(\log n)$ re-colorings. Explain the relevance of this result in terms of the theorem.
- (b) Examine Figures 13.4, 13.5, and 13.6 (pages 317, 320, and 321, respectively) in CLRS (3rd edition). Some of the cases handled by the main loop of the code of RB-INSERT-FIXUP are terminating—once encountered, they cause the loop to terminate after a small constant number of operations. For each of the cases of RB-INSERT-FIXUP, specify which are terminating and which are not.
- (c) Let T' be the result of applying Case 1 of RB-INSERT-FIXUP to T and define the potential function

$$\Phi(T) = \text{number of red nodes in } T.$$

Prove that $\Phi(T') = \Phi(T) - 1$.

- (d) Node insertion into a red-black tree using RB-INSERT can be broken down into three parts, TREE-INSERT, the non-terminating cases of RB-INSERT-FIXUP, and the terminating cases of RB-INSERT-FIXUP. Give the structural modifications and changes in potential resulting from these three parts.
- (e) Using the previous part, prove that the amortized number of structural modifications of RB-INSERT is $O(1)$.