PhD Qualifier Exam for Fall 2006—— Theory Area

CS Department, IIT

Your random number________________________________

There are 4 questions in this exam. For every question, please write your answer in a clean and concise way.

If you are asked to write an algorithm for a question, you have to write the **pseudo-code** of your algorithm and also put explanation about your pseudo-code.

1. The Fibonacci Numbers F_n, for $n \geq 0$ is defined as follows. $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 3$$

   Assume that, given any two integers x and y, we can compute their summ $x + y$ and their product $x \cdot y$ in a constant time. Design a method to compute the value of F_n in time $\Theta(\log n)$. You need to give a pseudocode of your method and also justify that you algorithm indeed runs in time $\Theta(\log n)$.

   Hint: Notice the following formula is true for Fibonacci numbers, when $n \geq 2$:

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}$$

2. Assume that you *only* know the following problems are NP-complete: SAT, 3SAT, VERTEX-COVER, CLIQUE, HAM-CYCLE, SUBSET-SUM (for the definition of the problems, look at the included chapter of Cormen et. al)

   Consider the following decision problem, called MINIMUM WEIGHT STRONGLY CONNECTED SPANNING SUBGRAPH: Given a directed graph $G = (V, A)$ with non-negative integer weights w on the acs of A, and an integer K, is there a set of arcs $B \subseteq A$ such that the graph (V, B) is strongly connected and $\sum_{e \in B} w(e) \leq K$.

   Prove that the MINIMUM WEIGHT STRONGLY CONNECTED SPANNING SUBGRAPH problem is NP-complete.

3. Give a polynomial-time algorithm for computing a maximum-weight independent set in a tree. The formal definition of the problem is: Given a tree $T = (V, E)$ whose vertices have weights w, find a maximum weight set $S \subseteq V$ such that no two elements of S are adjacent in T. The weight of a set $A \subseteq V$ is $\sum_{v \in A} w(v)$.

   Assume the tree is represented as follows: for each node you have two, possibly NULL, links: one to the leftmost child, and one to the next sibling to the right.

4. Suppose that you have a “black-box” worst-case linear-time median subroutine. Give a simple, linear-time algorithm that, given an array $A[1..n]$ and a positive integer $i \leq n$ finds the i^{th} smallest element of A.